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A NEW CANADIAN ALGEBRA

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EDITORS' NOTE

THIS is a revision of Durell and Wright's *Elementary Algebra, Part II.*, published in Great Britain by G. Bell & Sons, Ltd. It is issued in this form with the permission of the English authors and publishers. Our aim has been to provide a modern text in Algebra for use in the senior year of Canadian secondary schools. It is assumed that all the students using this book will have had at least two years of Algebra, the usual elementary Plane Geometry, and the rudiments of elementary Science. Many students, it is further assumed, will combine with this course Plane Trigonometry. They are indeed fortunate who have the opportunity to compare the functions here studied with the periodic functions used throughout physical science.

That elementary notions of *variable*, *function*, and *rate of change* are an almost indispensable element in the curriculum of secondary schools has been a growing conviction among educationists. The *derivative* has long been employed as a tool in Algebra, but its simplicity and usefulness in elementary scientific and mathematical work were not at first fully realised. For non-specialist and future specialist alike, the study by analytical and graphical methods of linear, quadratic, power, and growth functions has come to be regarded as fundamental and more than worthy of a place beside the binomial theorem and the conic sections.

E.W.S.

D.L.S.

May, 1933.

AUTHORS' PREFACE

THE general lines of procedure adopted in *Elementary Algebra, Parts I. and II.*, have been followed in this volume. General discussions in the text have been avoided wherever possible, methods have been indicated by illustrative examples, the exercises have been arranged to lead the pupil to construct his own formulæ, and summaries of results, established in the exercises, have been added to consolidate the progress made and to assist revision.

In selecting their material, the authors have chiefly kept in mind the needs of the non-specialist, and have derived much assistance from the recommendations made in the *Reports of the Mathematical Association*. Every effort has been made to introduce as much variety as possible into the examples, and to infuse them with practical interest. Repetitions of special types may eventually induce some kind of facility in handling them, but this method dulls the mind ; it is variety alone which makes the pupil " keep thinking."

Acknowledgment is due to the Controller of H.M. Stationery Office and to the Oxford and Cambridge Joint Board for kind permission to include questions set in examinations, and to Messrs. Macmillan & Co., Ltd., who have sanctioned the use of their method of arrangement of logarithm tables.

C.V.D.
R.M.W.

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CHAPTER I.

RATIO AND PROPORTION. VARIATION.

THE *ratio* of two numbers a, b is measured by the fraction $\frac{a}{b}$ and is often written $a : b$.

The ratio $na : nb$ equals the ratio $a : b$, because $\frac{na}{nb} = \frac{a}{b}$.

The ratio-relation $x : y = a : b$ may be written either as

$$\frac{x}{y} = \frac{a}{b} \text{ or } \frac{x}{a} = \frac{y}{b}.$$

Such ratio-relations as $x : y : z : u : v \dots = a : b : c : d : e \dots$ may be written either as

$$\frac{x}{y} = \frac{a}{b} \text{ and } \frac{y}{z} = \frac{b}{c} \text{ and } \frac{z}{u} = \frac{c}{d}, \text{ etc.,}$$

or as
$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{u}{d} = \dots$$

Example I. Simplify the ratio $4\frac{1}{5} : 3\frac{1}{2}$.

$$\frac{4\frac{1}{5}}{3\frac{1}{2}} = \frac{4\frac{1}{5} \times 10}{3\frac{1}{2} \times 10} = \frac{42}{35} = \frac{6}{5};$$

$\therefore$ the ratio equals $6 : 5$.

Example II. If $x : y = 4 : 7$, find the value of the ratio $(x + 2y) : (4x - y)$.

First Method.

Suppose $x = 4a$, then $y = 7a$;

$$\therefore \frac{x + 2y}{4x - y} = \frac{4a + 14a}{16a - 7a} = \frac{18a}{9a} = \frac{2}{1}.$$

$\therefore$ the ratio equals $2 : 1$ or 2 .

Second Method.

$$\frac{x + 2y}{4x - y} = \frac{\frac{x}{y} + 2}{\frac{4x}{y} - 1} = \frac{\frac{4}{7} + 2}{\frac{16}{7} - 1} = \frac{18}{7} \div \frac{9}{7} = 2.$$

Example III. Given

$x : y = 2 : 3$ and $y : z = 5 : 7$ and $z : w = 3 : 2$,
express in its simplest form $x : y : z : w$.

$$\begin{aligned} \frac{x}{2} &= \frac{y}{3} \quad \text{and} \quad \frac{y}{5} = \frac{z}{7} \quad \text{and} \quad \frac{z}{3} = \frac{w}{2}; \\ \therefore \frac{x}{10} &= \frac{y}{15} \quad \text{and} \quad \frac{y}{15} = \frac{z}{21} \quad \text{and} \quad \frac{z}{21} = \frac{w}{14}; \\ \therefore \frac{x}{10} &= \frac{y}{15} = \frac{z}{21} = \frac{w}{14}; \\ \therefore x : y : z : w &= 10 : 15 : 21 : 14. \end{aligned}$$

EXERCISE I. a.

1. Simplify the following ratios :

✓(i) $3\frac{1}{3} : 1\frac{1}{4}$;

(ii) $a : (a + \frac{1}{2}a)$;

(iii) $\frac{a}{b} : ab$;

(iv) $(1 + \frac{1}{x}) : (1 + x)$;

✓(v) $15x^2y : 12xy^2$;

(vi) $10^2 : 20^2$;

(vii) $10^6 : 10^3$;

✓(viii) $(a^2 - b^2) : (a^2 - ab)$;

(ix) a yards : b feet ;

(x) x dollars : y cents.

✓2. If $3x + 5y = 7x - 13y$, find $x : y$.

✓3. If $a : b = 3 : 5$, evaluate $(a + b) : (a - b)$.

4. If $\frac{x+11}{y+7} = \frac{11}{7}$, prove $\frac{x}{y} = \frac{11}{7}$.

5. Express in ratio form the following equations :

- (i) $ab=cd$; (ii) $p^2=qr$; (iii) $(a+b)(c+d)=(p+q)(e+f)$;
 (iv) $xyz=pqr$; (v) $abc=de$.

6. Two numbers are in the ratio $a : b$; the first is x ; what is the second ?

7. y is the result of increasing x by 10 per cent. ; find $x : y$.

8. If a is x per cent. greater than b , by what per cent. is b less than a ?

9. A man's salary was $\$x$ before the war and has been increased since the war by $\$y$. In what ratio has it been altered ?

10. $ABCD$ is a straight line ; $AB=x''$, $BC=2x''$, $CD=4x''$; what is the ratio of AC to BD ?

11. The rents of two houses used to be in the ratio $a : b$; but the first has been raised 20 per cent. and the second 50 per cent. ; what is their present ratio ?

12. Two partners share their profits in the ratio $a : b$; the profits are $\$p$; how much does each receive ?

13. The ratio of the areas of two triangles is $a : b$ and the ratio of their heights is $p : q$; what is the ratio of their bases ?

14. (i) Find two numbers in the ratio $7 : 5$, whose difference is 8.

(ii) Find two numbers in the ratio $p : q$, whose difference is c .

15. The line AB is divided externally at P in the ratio $5 : 2$; $AB = x''$; find AP .

16. The ratio of the weights of two circular cylinders of the same material is $3 : 2$ and the ratio of their diameters is $3 : 4$; find the ratio of their heights.

17. AB is divided externally at P in the ratio $3 : 5$ and externally at Q in the ratio $9 : 7$; find $PQ : AB$.

18. A measuring rod contracts in the ratio $a : b$ where $a < b$; it is used to measure a line whose real length is x inches ; what is the length obtained and what is the error per cent. ?

19. If x is positive, is the ratio $(3 + x) : (5 + x)$ greater or less than the ratio $3 : 5$?

20. If x is positive, is the ratio $(8 + x) : (7 + x)$ greater or less than the ratio $8 : 7$?

21. If $\frac{1}{a} + \frac{1}{b} = 3\left(\frac{1}{a} - \frac{1}{b}\right)$, find $a : b$.

22. If $3x - y = 2z$ and $y = 6(z - x)$, find $x : y : z$.

23. A chain of length l feet hangs over a small rough peg ; the portions on each side are in the ratio $a : b$; if $a > b$, find the depth of the mid-point of the chain below the peg.

24. $ABCD$ is a straight line ; B divides AC in the ratio $p : q$; C divides BD in the ratio $x : y$; find (i) $AB : CD$, (ii) $AB : BD$.

25. If $(8x - 2y)^2 = (3x + 4y)^2$, find $x : y$.

26. If $4x^2 + 9y^2 = 12xy$, find $x : y$.

27. The perimeter of a right-angled triangle is four times the shortest side ; find the ratio of the other two sides.

28. The ratio of the radii of two circles is $a : b$ and the ratio of their areas is $(a - x) : (b - x)$; express x in terms of a, b .

29. If $\frac{x + y}{x - y} = \frac{7}{4}$, evaluate $\frac{x^2 + y^2}{x^2 - y^2}$.

30. If x and y are each increased in the ratio $a : b$, in what ratio is $\frac{x + y}{x - y}$ altered ?

31. At a height of h feet it is possible to see a distance of $\sqrt{\frac{3h}{2}}$ miles ; an aeroplane climbs from 4000 feet to 9000 feet ; in what ratio does the pilot increase his range of view ?

32. The horse-power H required to drive W lb. of water per second through a small circular aperture of area A sq. inches is given by $H = 0.00015 \frac{W^3}{A^2}$. In what ratio must the horse-power be altered if the radius of the aperture is doubled and if twice as much water per second is required ?

33. The indicated horse-power H required to produce a speed of V knots in a paddle steamer is given by the approximate formula $H = \frac{A \cdot V^3}{70}$, where A is its greatest section in sq. yards. (i) In what ratio must the horse-power be increased to raise the speed from 8 to 12 knots? (ii) The values of A for two steamers are in the ratio 6 : 5 and their speeds are in the ratio 5 : 2; what is the ratio of their indicated horse-powers?

34. If a carrier is driven through a tube l feet long, d feet in diameter, under a pressure of P lb. per sq. inch, the time of transmission is approximately $0.0048 \sqrt{\frac{l^3}{Pd}}$ seconds; what is the effect on the time of transmission if the length is increased in the ratio 3 : 2, the pressure in the ratio 5 : 2, and the diameter decreased in the ratio 3 : 5?

35. If $a > b > x > 0$, arrange the ratios $(a - x) : (b - x)$ and $(a + x) : (b + x)$ and $a : b$ in ascending order of magnitude.

36. The ratio of a man's expenditure to his income is $a : b$; what is the ratio of his savings to his expenditure?

37. The ratio of the male to the female voters at an election is $a : b$. If c fewer men and d fewer women had voted, the ratio would have been $p : q$. What was the total number of votes cast in terms of these letters?

38. If h is small compared with 1000, find an approximate value of the ratio $[(1000 + h)^2 - 1000^2] : h$.

39. If h is small compared with x and y , find an approximate value of the ratio $[(x + h)^2 - x^2] : [(y + h)^2 - y^2]$.

40. The incomes of A and B are in the ratio 5 : 4; their expenditures are in the ratio 6 : 5 and their savings are in the ratio 10 : 7. Find the ratio of A 's income to B 's expenditure.

If $\frac{a}{b} = \frac{c}{d}$, the numbers a, b, c, d are said to be *in proportion*, and d is called the *fourth proportional* to a, b, c .

If $\frac{a}{b} = \frac{b}{c}$, c is called the *third proportional* to a, b , and b is called the *mean proportional* between a, c .

If $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} = \text{etc.}$, $a, b, c, d, e, \dots$ are said to be in *continued proportion*.

The ratio-equality $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{w}{d} = \dots$ is often written in the form $x : y : z : w : \dots = a : b : c : d : \dots$

Example IV. If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{a}{b} = \frac{a+c}{b+d} = \frac{a-c}{b-d}$.

Let $\frac{a}{b} = \frac{c}{d} = k$;

$$\therefore a = bk; \quad c = dk;$$

$$\therefore \frac{a+c}{b+d} = \frac{bk+dk}{b+d} = \frac{k(b+d)}{b+d} = k = \frac{a}{b}$$

and $\frac{a-c}{b-d} = \frac{bk-dk}{b-d} = k = \frac{a}{b}.$

EXERCISE I. b.

1. Find the fourth proportional to

(i) 2, 3, 4; (ii) ab, bc, cd ; (iii) $\frac{1}{x}, y, \frac{1}{y}$

2. Find the third proportional to

(i) 2, 3; (ii) ab, bc ; (iii) $\frac{1}{x}, y$.

3. Find a mean proportional between

(i) 3, 12; (ii) a^3b, ab^3 ; (iii) a, b .

4. Fill up the gaps in

$$\frac{x}{8} = \frac{y}{3} = \frac{x+y}{3} = \frac{x-y}{3} = \frac{5x-12y}{3}.$$

5. If $x : y : z = 3 : 2 : 7$, find $\frac{x + 3y}{y + z}$ and the ratios $\left[\frac{1}{x} : \frac{1}{y} : \frac{1}{z} \right] ?$

6. If $\frac{x + y + z}{11} = \frac{x + y - z}{8} = \frac{x - y}{5}$, find $x : y : z$.

7. If $\frac{x}{5} = \frac{y + z}{11} = \frac{x - y + z}{2}$, find $x : y : z$.

8. If $\frac{a}{b} = \frac{c}{d}$, prove $\frac{a}{b} = \frac{\sqrt{a^2 + c^2}}{\sqrt{b^2 + d^2}}$.

9. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove $\frac{3a - 5c}{3b - 5d} = \frac{2c + 7e}{2d + 7f}$.

10. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, fill up the gaps in

$$\frac{a}{b} = \frac{a + c + e}{b - d + f} = \frac{10a - 7c + 2e}{b - d + f}.$$

11. X, Y are points on the sides AB, AC of the triangle ABC such that $\frac{AX}{XB} = \frac{AY}{YC}$; prove that (i) $\frac{AX}{AB} = \frac{AY}{AC}$; *Inversion add (1)*
(ii) $\frac{BX}{AB} = \frac{CY}{AC}$. *add 1. & invert.*

12. O is a point inside the triangle ABC ; AO is produced to cut BC at D ; prove that (i) $\frac{\triangle BAD}{\triangle CAD} = \frac{BD}{CD}$; (ii) $\frac{\triangle BAO}{\triangle CAO} = \frac{BD}{CD}$.

13. If $\frac{a}{b} = \frac{c}{d}$, prove that

$$(i) \frac{a}{b} = \sqrt{\frac{a^2 + c^2}{b^2 + d^2}}; \quad (ii) \frac{a - c}{b - d} = \sqrt[5]{\frac{a^5 + c^5}{b^5 + d^5}}.$$

14. If b is a mean proportional between a, c , prove that $\frac{a^2}{b^2} = \frac{a}{c}$. Interpret this geometrically.

15. Given that z can be determined when x and y are known and that the values of z are shown in the following double-entry table :

		x			
		0	1	2	3
y	0	0	2	4	6
	1	1	4	9	16
	2	2	6	14	26
	3	3	8	19	36

Find the probable values of z when

- (i) $x = 1, y = 1.5$; (ii) $x = 2.5, y = 2$;
 (iii) $x = 1.5, y = 1.5$; (iv) $x = 4, y = 4$.

SUMMARY OF RESULTS.

If
$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \dots,$$

then each fraction $\equiv \frac{pa + qc + re + \dots}{pb + qd + rf + \dots}$
 $\equiv \sqrt[n]{\left\{ \frac{pa^n + qc^n + \dots}{pb^n + qd^n + \dots} \right\}}.$

VARIATION.

If a man is walking at a uniform rate of 4 miles an hour, he walks 1 mile in 15 minutes,

2 miles in 30 minutes,

3 miles in 45 minutes, and so on.

The distance he walks is *directly proportional* to the time taken.

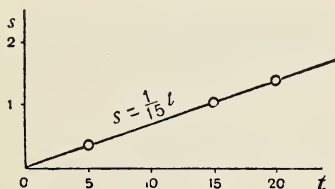
If he walks s miles in t minutes, we have

$$s = \frac{t}{15} \text{ or } \frac{s}{t} = \frac{1}{15}.$$

In this case, s is said to *vary directly as t* .

This is written $s \propto t$.

The graph of the equation $s = \frac{1}{15}t$ is a straight line through the origin. We plot the values of t as abscissæ and the corresponding values of s as ordinates, i.e. we "plot s against t ."



Suppose next there are a number of rectangles each of area 60 sq. in., but of different shapes.

If the breadth is 2", the length is 30".

If the breadth is 3", the length is 20".

If the breadth is 4", the length is 15", and so on.

The breadth *is inversely proportional to the length*.

If the breadth is b in. and the length is l in.,

$$b = \frac{30}{l} \text{ or } \frac{b}{1} = \frac{30}{l}.$$

In this case, b is said to *vary inversely as l* .

This is the same as saying that b varies directly as $\frac{1}{l}$ or

$$b \propto \frac{1}{l}.$$

And the result of plotting b against $\frac{1}{l}$ is a straight line through the origin.

Conversely, if we are given a table of values for x and y , and if the graph is a straight line through the origin when y is plotted against x , then $y \propto x$.

If the graph is a straight line through the origin when y is plotted against $\frac{1}{x}$, then $y \propto \frac{1}{x}$.

Example V. The weights of a number of circular discs of the same material and thickness but different diameters are given in the following table :

Diam. in cm. = $d = 2$	3	4	5	6	7	8	9	10
Weight in gr. = $w = 20$	45	80	125	180	245	320	405	500

Show that the weight varies as the square of the diameter, and plot w against d^2 .

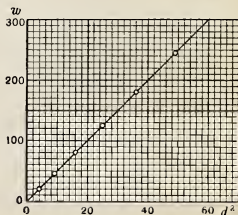
The table connecting d^2 and w is

d^2	4	9	16	25	36	49	64	81	100
w	20	45	80	125	180	245	320	405	500

from which we see that

$$w = 5d^2.$$

$$\therefore w \propto d^2.$$



The result of plotting w against d^2 gives a straight-line graph passing through the origin, which is the graphical test that w is proportional to d^2 . The graph of w against d should now be drawn, and compared with the graph of w against d^2 .

Example VI. It is stated that if the velocity of a stream of water is determined by a water-pressure gauge, the velocity varies as the square root of the height in the gauge.

(i) If a speed of 8 feet a second causes a height of 12 inches, what is the speed when the recorded height is 4 inches?

(ii) If the velocity is v feet per second when the height recorded is h inches, find the equation connecting v and h .

(i) We have $v \propto \sqrt{h}$.

Now $v = 8$, $h = 12$ is one pair of corresponding values; and it is required to find the value of v which corresponds to $h = 4$.

$\therefore$ when $h = 4$, the value of v is given by

$$\frac{v}{8} = \frac{\sqrt{4}}{\sqrt{12}};$$

$$\therefore v = \frac{8\sqrt{4}}{\sqrt{12}};$$

$$\therefore v = 4.62 \text{ feet per sec.}$$

(ii) Also we have

$$\frac{v}{8} = \frac{\sqrt{h}}{\sqrt{12}};$$

$$\therefore v = \frac{8}{\sqrt{12}} \cdot \sqrt{h};$$

$$\therefore v = 2.31 \cdot \sqrt{h} \text{ approximately.}$$

EXERCISE I. c.

1. A man walking 3 miles an hour travels y yards in x minutes :

- (i) Does y vary directly as x ?
- (ii) What is the effect on y of doubling x ?
- (iii) What is the effect on x of doubling y ?
- (iv) What equation connects x and y , and sketch its graph ?

2. x men take y days to level a large field which it would take 12 men 40 days to level :

- (i) Does y vary directly as x ?
- (ii) What is the effect on y of doubling x ?
- (iii) What is the effect on x of halving y ?
- (iv) Express y in terms of x , and sketch its graph.

3. If y varies as x , (i) what is the effect on y of trebling x ? (ii) what is the effect on x of halving y ? (iii) what is the shape of the graph of y ?

4. If y varies inversely as x , (i) what is the effect on y of doubling x ? (ii) what is the effect on x of multiplying y by 10 ? (iii) what is the shape of the graph of y ? (iv) if x is the length and y the height of a rectangle, what can you say about the area of the rectangle ?

5. A rectangle of area 60 sq. cm. is x cm. long and y cm. wide.

- (i) What is the variation-relation between y and x ?
- (ii) Complete the table :

y	10	20	30	40	50
x	6	3	2	$\frac{3}{2}$	$\frac{6}{5}$
$\frac{1}{x}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$

- (iii) Plot $\frac{1}{x}$ against y , taking 3" as the unit on the $\frac{1}{x}$ axis.

6. If y varies as x^2 , what is the effect (i) on x of dividing y by 4 ; (ii) on y of dividing x by 4 ? What is the shape of the graph of y ?

7. Construct a table of values of x and y to illustrate the cases where

- (i) y varies as x ;
- (ii) y varies inversely as x ;
- (iii) y varies as x^3 .

Sketch the graph of y in each case.

8. If multiplying x by 10 always produces the effect of dividing y by 100, what is the variation-relation between x and y ?

9. If $y = x + x^2$, in what ratio is y increased when x alters from (i) 1 to 2, (ii) 2 to 4, (iii) 4 to 8? Does y vary as any power of x ?

10. In the following tables y varies directly or inversely as some power of x ; find the relation between y and x in each case. What method of plotting must be adopted to give straight-line graphs in the various cases? Carry it out in each case.

(i)	$x =$	— 3	— 1	0	2	5	
	$y =$	— 12	— 4	0	8	20	
(ii)	$x =$	1	2	3	4	5	
	$y =$	2	8	18	32	50	
(iii)	$x =$	1	2	5	8	10	12
	$y =$	10	5	2	$1\frac{1}{4}$	1	$\frac{5}{6}$
(iv)	$x =$	1	4	9	25	36	
	$y =$	1	2	3	5	6	

11. A number of triangles each have a base 6 cm. long; if the height of any one of them is h cm. and its area is A sq. cm., complete the following table; what is the variation between A and h ?

$h = 3$	4	5	8	10
$A = 9$	12	15*	24	30.

12. If y varies as x^2 , complete the following table:

$x = 0$	1	2	3	5	10
$y =$		12			

13. If y varies as $\frac{1}{x}$, complete the table :

$x = 1$	4	10	20
$y = 8$	2	0.8	.4

14. A cube of aluminium whose edge is x cm. weighs y gr. ; complete the table :

$x = 1$	2	4	5
$y = 26$	20.8	166.4	325

15. If y varies as $\sqrt{x}$, complete the table :

$x = 1$	1	2	4	25	100
$y = 1$	2	2.5	4	10	20

16. In the given graph, Fig. 1, $y \propto x^2$; find the relation between x and y .

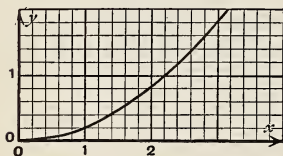


FIG. 1.

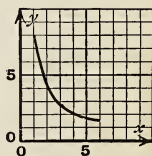


FIG. 2.

17. In the given graph, Fig. 2, y varies inversely as x ; find the relation between x and y .

18. If y varies as x and if $y = 3$ when $x = 2$, represent the relation between y and x by a graph.

19. The area of the surface S sq. in. of a sphere varies as the square of its radius r in. ; its volume V cu. in. varies as

the cube of its radius: (i) what is the effect on V of increasing S in the ratio $64:1$? (ii) what is the variation-relation between S and V ?

20. The pressure on the wind-shield of a car travelling v miles an hour on a still day is P lb. per sq. foot. It is found that $P \propto v^2$. Is this consistent with the following measurements?

$v = 5$	20	30
$P = 0.123$	1.97	4.43

Express P in terms of v as accurately as the data permit.

21. The range of view at sea varies as the square root of the height of the observer above sea-level. At a height of 20 feet the look-out can see 5.5 nautical miles; what is the range of vision at a height of 40 feet? If a height of h feet commands N nautical miles, express N in terms of h .

22. Two hot-water cans are the same shape: one is 8" high and holds a quart, the other is 2' high. What does it hold?

23. Two candlesticks are the same shape; it costs 15 shillings to gild the smaller which is 6" high; what is the cost of gilding the other which is 15" high?

24. In what ratio must the radius of a sphere be increased to double (i) its surface, (ii) its volume?

25. If a statuette 12" high, made of bronze, weighs 4 lb. what is the weight of a plaster model 3" high, bronze being 10.7 times as heavy as plaster?

26. It takes x lb. of paint to paint the hull of a battleship 400 feet long; how much will be required for the hull, similar in shape, of a battleship 530 feet long?

1.33x
lbs.

27. The weights of two spheres are in the ratio 2:1 and the densities of the materials of which they are made are in the ratio 1:2. Compare the radii.

$\frac{r_1}{r_2} = \frac{3\sqrt{2}}{1}$

28. The time of a complete oscillation of a pendulum varies as the square root of its length. If a pendulum of length 100 cm. makes one complete oscillation in 2 seconds, find the

time of a complete oscillation of a pendulum of length 25 cm. If the time is t sec. when the length is l cm., express t in terms of l .

SUMMARY OF RESULTS.

y is said to be a "function" of x , if y can be determined when x is known.

Suppose that y is a function of x , and that the values $y_1, y_2, y_3, y_4, \dots$ of y correspond to the values $x_1, x_2, x_3, x_4, \dots$ respectively of x . If the ratios $y_1 : y_2 : y_3 : y_4$, etc., are respectively equal to the ratios $x_1 : x_2 : x_3 : x_4$, etc., then y is said to *vary directly* as x , or simply *vary* as x .

This is sometimes written $y \propto x$.

If the ratios $y_1 : y_2 : y_3 : y_4$, etc., are equal to the ratios $x_1^n : x_2^n : x_3^n : x_4^n$, etc., then y is said to *vary directly as the n th power of x* , or $y \propto x^n$.

If the ratios $y_1 : y_2 : y_3 : y_4$, etc., are equal to the ratios $\frac{1}{x_1^n} : \frac{1}{x_2^n} : \frac{1}{x_3^n} : \frac{1}{x_4^n}$, etc., then y is said to *vary inversely as the n th power of x* , or $y \propto \frac{1}{x^n}$.

The statement that y varies as x^n may be expressed in any of the following ways :

(i) $y \propto x^n$.

(ii) $\frac{y_p}{y_q} = \frac{x_p^n}{x_q^n}$, where x_p, y_p are any pair of corresponding values of x and y , and x_q, y_q are any other pair.

(iii) $\frac{y_1}{x_1^n} = \frac{y_2}{x_2^n} = \frac{y_3}{x_3^n} = \dots = k$ (say).

(iv) $y = k \cdot x^n$, where k is a *constant* number independent of the values of x and y .

(v) If x is altered in any ratio $\lambda : 1$, then y is altered in the ratio $\lambda^n : 1$.

(vi) The result of plotting y against x^n gives a straight line through the origin.

JOINT VARIATION.

First suppose we have a number of circular cylinders each of radius 5 in., but of different heights.

If the volume of one of these cylinders of height h in. is V cu. in., then

$$V = 25\pi h.$$

For this set of cylinders, $V \propto h$.

Next suppose we have a number of circular cylinders, each of height 7 in., but of different radii.

If the volume of one of these cylinders of radius r in. is V cu. in., then

$$V = 7\pi r^2.$$

For this set of cylinders, $V \propto r^2$.

These facts can be expressed by saying that

$$V \propto h \text{ when } r \text{ is constant}$$

and

$$V \propto r^2 \text{ when } h \text{ is constant.}$$

Lastly, if both r and h vary, so that the cylinder may be of any height and radius,

$$V = \pi r^2 h$$

or

$$V \propto r^2 h \text{ when both } r \text{ and } h \text{ vary.}$$

This example illustrates the general statement that :

if x varies as y when z is constant

and x varies as z when y is constant,

then x varies as yz when y and z both vary,

or x is said to vary jointly as y and z .

Example VII. The force necessary to stop a train in a given distance varies directly as the weight of the train and the square of its velocity and inversely as the distance. A force of 10 tons will stop a train weighing 200 tons and travelling 30 miles an hour in 200 yards. Find the formula

which expresses the force F tons in terms of the velocity, v m.p.h., the distance, d feet, for a train weighing W tons.

$$F \propto \frac{Wv^2}{d}.$$

When $W = 200$, $v = 30$, $d = 600$, we have $F = 10$;

$$\therefore F : \frac{Wv^2}{d} = 10 : \frac{200 \times 30^2}{600} = 10 : 300 = 1 : 30 ;$$

$$\therefore F = \frac{Wv^2}{30d}.$$

Note.—The statement $F \propto \frac{Wv^2}{d}$ can be expressed as follows :

$$F = k \cdot \frac{Wv^2}{d}, \text{ where } k \text{ is a constant.}$$

In the above example, $k = \frac{1}{30}$.

EXERCISE I. d.

1. The volume, V cu. in., of a circular cone of height 6 in. and base-radius r in. is given by $V = 2\pi r^2$. The volume, V cu. in., of a circular cone of base-radius 6 in. and height h in. is given by $V = 12\pi h$. For a circular cone base-radius r in., height h , volume V cu. in.,

- (i) state how V varies if r is constant ;
- (ii) state how V varies if h is constant ;
- (iii) state how V varies if r and h both vary ;
- (iv) find the formula for V in terms of r and h .

2. One end of a cast-iron beam of rectangular section is built into a wall. For a beam 4" deep, 10' long, the greatest weight W tons that can be suspended from the other end is given by $W = 0.3b$ where b in. is its breadth. For a beam d " deep, 10' long, 8" broad, the greatest weight W tons is given by $W = 0.15d^2$.

For a beam 2" deep, l feet long, 4" broad, the greatest weight W tons is given by $W = \frac{3}{l}$.

State how W varies

- (i) if d, l are constant ;
- (ii) if b, l are constant ;
- (iii) if b, d are constant ;
- (iv) if b, d, l all vary.

Find the formula for W in terms of b, d, l .

3. A $\frac{1}{4}$ -lb. weight is fastened to the end of a string, and is whirled round in a circle. If the speed of the weight is 8 feet a second and the length of the string is l in., the strain in the string is T lb., where $T = \frac{6}{l}$. If the speed of the weight is v feet a second and the length of the string is 6 in., the strain is T lb., where $T = \frac{v^2}{64}$.

State how T varies

- (i) if v is constant ;
- (ii) if l is constant ;
- (iii) if v and l both vary.

Find the formula for T in terms of v and l .

If the string breaks under a strain of 5 lb., what is the maximum speed for a string of length 15 in. ? How does v vary if T is constant ?

4. A mass W lb. is fastened to the end of a string l feet long and is whirled round at the rate of n revolutions per sec. ; the strain in the string is T lb.

$$\text{If } W = 1, \quad n = 4, \quad \text{then } T = 2\pi^2 l.$$

$$\text{If } W = 2, \quad l = 4, \quad \text{then } T = \pi^2 n^2.$$

$$\text{If } l = 2, \quad n = 6, \quad \text{then } T = 9\pi^2 W.$$

- (i) How does T vary if l, n, W all vary ?
- (ii) Find the formula for T in terms of l, n, W .
- (iii) How does n vary if T and W are constant ?
- (iv) How does l vary if W alone is constant ?
- (v) What is the effect on T of doubling n , halving l and leaving W unchanged ?

5. p varies as t if v is constant, and varies inversely as v if t is constant. Complete the given double-entry table, giving the values of p .

$$p \propto \frac{t}{v}$$

$$p = k \frac{t}{v}$$

Values
of v

	Values of t .		
	300	350	400
40			
50			
60			240

Given $p = 240$
 $t = 300$
 $v = 60$
 value of const.

6. V varies as x if y is constant and varies as y^2 if x is constant; complete the given double-entry table, giving values of V .

Values of x .

	Values of x .		
	5	10	15
1			
2		24	
3			

$V = kx^2$
 $v = 3$

7. $S = \frac{6}{x}$ if $y = 2$, and $S = \frac{8}{y^2}$ if $x = 3$; what is the simplest expression for S in terms of x and y ?

8. Write down relations expressing the following facts:

- The area of the curved surface of a cone varies as the length of the slant side if the base-radius is constant, and varies as the base-radius if the length of the slant side is constant. [s, l, r.]
- The pressure of a gas varies directly as the absolute temperature if the volume remains constant, and varies inversely as the volume if the temperature remains constant. [p, T, v.]
- The time taken over a journey varies directly as the distance and inversely as the speed (uniform). [t, d, v.]

- (iv) The pressure required to drive x cu. ft. of gas per hour through a pipe varies directly as the length of the pipe and as the square of x , and inversely as the fifth power of the diameter. [p, l, d.]
- (v) The square of the time for pneumatic transmission through a tube varies directly as the cube of the length of the tube, and inversely as its diameter and the pressure. [t, l, d, p.]

9. On a railway curve the outer rail is raised above the inner by an amount which varies directly as the gauge and the square of the maximum velocity permitted, and inversely as the radius of the curve. If the gauge is 5 feet, the speed 15 m.p.h. and the radius 200 yards, the elevation is 1.5 inches. Express the elevation d inches in terms of the gauge W feet, the speed v m.p.h., and the radius R feet.

✓ 10. Lord Kelvin states that the most economical diameter for a copper wire in an electric circuit varies directly as the square root of the normal current and the fourth root of the cost per horse-power-year, and inversely as the fourth root of the price of copper per lb. Express this by an equation. In what ratio should the diameter be altered if the cost of horse-power rises 50 per cent. and the price of copper rises 200 per cent., and the current is reduced by 25 per cent.?

11. In steamships of a certain type, if the displacement is d tons and if the indicated horse-power is H for a speed of v knots, the cube of H varies directly as the square of d and the ninth power of v . For a displacement of 1600 tons, the I.H.P. is 690 when the speed is 10 knots. Obtain a general formula.

12. The cost of the coal consumed by a steamer travelling at a steady speed varies directly as the distance and as the square of the speed. What relation connects the cost, the speed and the time taken for the journey? If the coal consumed for a journey of 200 sea miles at a speed of 10 knots cost \$400, find the equation connecting the cost, \$ C , the distance, d sea miles, and the time, T hours.

13. x varies inversely as y and z ; y varies directly as the square root of x and the square root of w . If x increases in the ratio 3 : 2 and z in the ratio 4 : 3, how does w alter?

*x increases ratio 8:3
z increases ratio 4:3*

SUMMARY OF RESULTS.

If x varies directly as the p th power of y and inversely as the q th power of z , then

$$x = k \cdot \frac{y^p}{z^q},$$

where k is a constant, independent of x, y, z .

If $x = a$, when $y = b$ and $z = c$, k is found from the equation

$$a = k \cdot \frac{b^p}{c^q}$$

or
$$\frac{x}{a} = \frac{y^p}{z^q} \div \frac{b^p}{c^q}, \quad \text{or} \quad x = \frac{ac^q}{b^p} \cdot \frac{y^p}{z^q}.$$

If y is altered in the ratio $\lambda : 1$, and if z is altered in the ratio $\mu : 1$, then

$$x \text{ is altered in the ratio } \frac{\lambda^p}{\mu^q} : 1.$$

CHAPTER II.

FUNCTIONS OF ONE VARIABLE.

A. REPRESENTATION OF FUNCTIONS.

If the area of a rectangle is fixed, the lengths of its sides cannot be chosen independently of one another ; the choice of a length for one side depends upon the length chosen for the other. In other words, "the length of one side is a *known function* of the length of the other, when the area is given."

For example, if the area is 36 sq. inches, and if one side is of length x inches, then the other side must be of length $\frac{36}{x}$ inches.

Example I. Sketch the graph of the function $\frac{36}{x}$.

An accurate drawing is not required and it is not necessary to make a table of values. All that is necessary is to note a few important features of the function as follows :

(i) If x is positive, $\frac{36}{x}$ is positive ;

if x is negative, $\frac{36}{x}$ is negative.

- (ii) If x is large and positive, $\frac{36}{x}$ is small, and by making x sufficiently large, we can make $\frac{36}{x}$ as small as we please.

In symbols, when $x \rightarrow +\infty$, $\frac{36}{x} \rightarrow 0$, which reads in words "when x increases without limit $\frac{36}{x}$ approaches zero as a limit."

- (iii) If x is small and positive, $\frac{36}{x}$ is large and positive

$$\left[\text{e.g. if } x = 0.1, \frac{36}{x} = 360 \right].$$

In symbols, when $x \rightarrow +0$, $\frac{36}{x} \rightarrow +\infty$.

- (iv) When $x = 0$, $\frac{36}{x}$ has no meaning.

- (v) For a negative value of x , the value of $\frac{36}{x}$ is equal in magnitude but opposite in sign to its value for the corresponding positive value of x .

And, when $x \rightarrow -0$, $\frac{36}{x} \rightarrow -\infty$, i.e. $\frac{36}{x}$ decreases without limit, and when $x \rightarrow -\infty$, i.e., when x decreases without limit, $\frac{36}{x} \rightarrow -0$.

The rough shape of the graph can now be sketched (see Fig. 3).

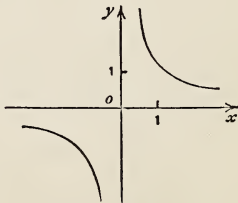


FIG. 3.

The graph is also said to correspond to the equation

$$y = \frac{36}{x} \quad \text{or} \quad xy = 36.$$

EXERCISE II. a.

[In the following examples, squared paper should not be used.]

1. Explain in words how the value of the function $(x-1)(x-3)$ varies as x varies from 1 to 3.

For what values of x does the function $= 0$?

For what range of values of x is the function positive ?

2. Can you find a value of x for which the function $x^2 - 4x + 4$ is negative ?

Construct another function of x which has a similar property.

3. For what values of x is the function $x^2 + x - 6$ zero ?

For what range of values of x is the function $x^2 + x - 6$ negative ?

4. (i) What is the greatest value of the function $9 - (x-1)^2$?

(ii) For what values of x is this function zero ?

(iii) What is the limit of this function when $x \rightarrow \infty$, and when $x \rightarrow -\infty$?

(iv) Sketch the graph of this function.

5. Sketch the graphs of (i) x^2 , (ii) $(x-3)^2$, (iii) $(x+2)^2$.

6. Sketch the graphs of

(i) $+\sqrt{x}$, (ii) $-\sqrt{x}$, (iii) $\pm\sqrt{x-2}$, (iv) $\pm\sqrt{x+1}$.

7. (i) Can you find (a) a positive value of x , (b) a negative value of x , such that $\frac{1}{x^2} < 0.01$?

(ii) When is this function $\frac{1}{x^2} > 100$?

(iii) Sketch the graph of $\frac{1}{x^2}$.

8. What is the value of the function $\frac{1}{x-3}$

(i) when x is large, e.g. $+1003$?

(ii) when x is large in absolute value, and negative, e.g. -997 ?

- (iii) when x is nearly equal to 3, (a) if $x > 3$, e.g. 3.001,
 (b) if $x < 3$, e.g. 2.999?

- (iv) Sketch the graph of $\frac{1}{x-3}$.

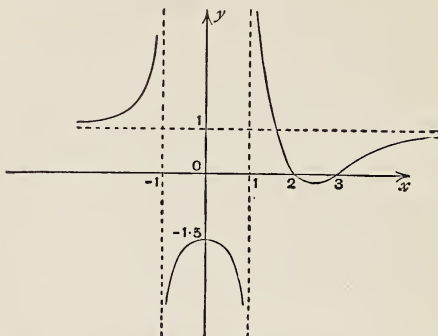


FIG. 4.

9. The graph of a certain function of x is shown in Fig. 4. Describe in words the variation in value of this function of x as x varies from $-\infty$ to $+\infty$.

10. ONP is a variable triangle with a right angle at N ; ON lies along the axis Ox ; the length of the hypotenuse OP is 5 and $ON = x$. What is the length of NP ?

As x varies, what is the locus of P ?

Of what function of x is this locus the graph?

11. The graph of a certain function of x is shown in Fig. 5. Describe in words how the value of the function varies as x varies from -2 to $+4$. If the same series of values of the function recurs again and again every time x is increased by 7, what can you say about the value of the function as $x \rightarrow \infty$?

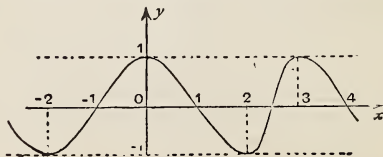


FIG. 5.

12. A certain function of x is defined for all values of x , positive, zero and negative. The function never has values greater than 1 or less than -2 , and is zero when x equals -1 or 3 or 4 . The function is positive if $-1 < x < 3$ or if $4 < x < \infty$, and for other values of x is negative.

Sketch the simplest graph of this function.

13. Sketch the simplest graph of a function which has the following characteristics :

- (i) $\rightarrow +1$, when $x \rightarrow +\infty$.
- (ii) $\rightarrow +\infty$, when $x \rightarrow +1$, provided $x > 1$.
- (iii) Is never equal to 1.
- (iv) Is not defined if $-1 < x < +1$.
- (v) $\rightarrow -1$, when $x \rightarrow -\infty$.
- (vi) $\rightarrow -\infty$, when $x \rightarrow -1$, provided $x < -1$.

Handwritten notes:
 $1 - \frac{1}{x}$
 $y = \frac{K}{x}$

14. (i) Can you find a value of x for which $\frac{x-1}{x-2}$ is firstly > 100 and secondly < -100 ?

(ii) For what value of x is this function zero?

(iii) For what range of values of x is this function negative?

(iv) Can you find a value of x for which the function equals 1?

(v) For what value of x is the function equal to 1.001?

(vi) For what value of x is the function equal to 0.999?

(vii) Sketch the graph of this function.

15. (i) For what values of x is the function $(x-1)(x-3)(x-4)$ zero?

(ii) For what range of values of x is this function positive?

(iii) Find the values of the function when $x = 10, 0, -10$.

(iv) Can you find a value of x for which the function is $> 1,000,000$?

(v) Sketch the graph of this function.

16. (i) Can you find a value of x between 0 and 10 for which the function

$$\frac{(x-1)(x-5)}{x-3}$$

is firstly > 1000 ; and secondly < -1000 ?

(ii) Can you find the value of the function when $x = 3$?

- (iii) Describe the changes of sign in the function as x increases from -1 to $+8$, stating also where it vanishes.
- (iv) What is the approximate error per cent. in taking the function as equal to x when $x = 1000$ and when $x = -1000$?
- (v) Sketch the graph of this function.

17. Sketch the graph of

- (i) $(x-1)(x-2)(x-3)(x-4)(x-5)$;
- (ii) $(x-1)(x-2)^2(x-4)(x-5)$;
- (iii) $(x-1)(x-2)^3(x-5)$.

18. (i) Find approximately the value of the function

$$\frac{x-1}{(x-2)(x-3)}$$

if $x = 1.99, 2.01, 2\frac{1}{2}, 2.99, 3.01$.

- (ii) What can you say about the value of this function when x is large; about how much, for example, is it if x is a million?
- (iii) What is its approximate value if $x = -1,000,000$?
- (iv) For what range of values of x is this function negative?
- (v) Sketch the graph of the function.

19. Sketch the graph of $\frac{x-2}{(x-1)(x-3)}$.

20. Sketch the graph of $\frac{x-1}{(x-2)^2}$.

21. Sketch the graph of $\frac{(x+1)(x-6)}{(x-1)(x-4)}$.

22. Sketch the graph of $\frac{x(x-3)}{(x+2)(x-1)}$.

SUMMARY OF RESULTS.

The function $f(x) \equiv \frac{a(x-b)}{c(x-d)}$ vanishes when $x = b$, and is not defined when $x = d$. In particular, the function $f(x) \equiv \frac{3(x-2)}{4(x-7)}$ vanishes when $x = 2$, is not defined when $x = 7$; and, further, the function increases without limit when x approaches 7 from the right, but decreases without limit when x approaches 7 from the left.

The statement that $f(x) \rightarrow +\infty$ when $x \rightarrow p$ means that $f(x)$ can be made to exceed any number however large, if x is given a value sufficiently close to p or any value closer than this to p .

In order to find the range of values for which a function is positive (or negative), it is usually best to factorise it, when possible.

B. CONSTRUCTION OF FUNCTIONS.

EXERCISE II. b.

In Fig. 6 the co-ordinates of A , B , P are respectively $(2, 3)$; $(6, 5)$; (x, y) . Express y as a function of x .

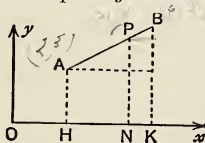


FIG. 6.

2. AB is the diameter of a circle APB ; PN is the perpendicular from P to AB ; if $AB = 8$, $AN = x$, $PN = y$, express y as a function of x .

3. $ABCD$ is a rectangle; P is a point such that the perpendicular PN from P to AB is equal to PC . If $BC = 4$, $PN = x$, $NB = y$, express y as a function of x .

4. PN is an altitude of the triangle APB ; O is the mid-point of AB . If $PN = y$, $ON = x$, $AB = 6$ and $PA^2 + PB^2 = 30$, express y as a function of x .

5. AB is a diameter of a circle; a line AQR cuts the circle at Q and the tangent at B in R ; P is a point on AQ such that $AP = QR$; PN, QK are the perpendiculars from P, Q to AB ; $AB = 4$, $AN = x$, $PN = y$, $QK = z$; express (i) z as a function of x (use the fact that $BK = AN$), (ii) y as a function of x .

Sketch the graph of y .

6. AOB is a triangle right-angled at O ; PN is the perpendicular from a point P on AB to OB ; if $AP = 3$, $PB = 4$, $ON = x$, $PN = y$, express y as a function of x .

7. A point R is taken on the side AB of a triangle ABC of area z sq. inches such that $AR = x \cdot AB$, where $x > \frac{1}{2}$. RQ, RH are drawn parallel to BC, AC to meet AC, BC at Q, H ; QK is drawn parallel to AB to meet BC at K . Express the area of $QRHK$ as a function of x and z .

8. AB is a diameter of a circle; CD is a chord parallel to AB and at distance b inches from it; any chord AQ cuts CD at R ; RN is drawn perpendicular to AB ; QP is drawn parallel to AB and cutting RN at P ; if $AB = a$, $AN = x$, $NP = y$ inches, express y as a function of x .

9. AB is a fixed diameter of a given circle; the tangent at B meets a variable chord AP at Q ; $AB = d$, $AP = x$, $BQ = y$; express y as a function of x .

10. $ABCD$ is a straight line and AH, BK, CL are three fixed lines perpendicular to it; $AB = a$, $BC = b$, $CD = c$; a variable line cuts AH, BK, CL at P, Q, R and DQ cuts CL at S ; if $AP = y$, $CR = z$, $CS = x$, express x as a function of y, z .

11. Assuming the length, breadth and depth of an ordinary match-box are in the ratio $10 : 7 : 3$, express the volume of the box as a function of the area of match-board used in making it, i.e. box and drawer.

12. Taking Ox, Oy as perpendicular axes, sketch the graphs of x^2 and $\frac{1}{x}$ for positive values of x ; any line perpendicular to Ox cuts the graphs at P, Q and Ox at N ; if $ON = x$, express the area of the triangle OPQ as a function of x .

13. A sector of a circle of unit radius is folded to form a circular cone. If the angle of the sector is x right angles, express the volume of the cone as a function of x , assuming $x < 4$.

14. Find a function of x which is zero for $x = 1$ and $x = 3$ and is positive if $1 < x < 3$.

15. Find a function of x which is equal to 2 when $x = 1$, but is never less than 2.

C. GRAPHICAL SOLUTIONS.

Example II. $ABCD$ is a rectangle such that $AB=4''$, $BC = 8''$; P, Q are points on BC, CD , such that

$$CQ = \frac{1}{2}BP = x \text{ inches.}$$

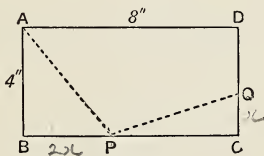


FIG. 7.

Express the area of $APQD$ in terms of x . Represent this function graphically: and find from the graph the length of CQ when the area of $APQD$ is 24 sq. inches.

Check the result by algebra.

(i) The area of $ABCD = 4 \times 8 = 32$ sq. inches.

$$CQ = x, \quad BP = 2x; \quad \therefore PC = 8 - 2x;$$

$$\therefore \text{area of triangle } ABP = \frac{1}{2} \times 4 \times 2x = 4x \text{ sq. inches.}$$

$$\text{area of triangle } PCQ = \frac{1}{2}(8 - 2x) \times x = x(4 - x) \text{ sq. in.};$$

$$\begin{aligned} \therefore \text{area of } APQD &= 32 - 4x - x(4 - x) \\ &= 32 - 4x - 4x + x^2 \\ &= 32 - 8x + x^2 \text{ sq. inches. } \textit{Answer.} \end{aligned}$$

(ii) To represent $x^2 - 8x + 32$ by a graph.

Since CQ does not exceed CD , x is not greater than 4.

Construct a table of values for x from 0 to 4.

x	0	1	2	3	4
x^2	0	1	4	9	16
$-8x$	0	-8	-16	-24	-32
32	32	32	32	32	32
$x^2 - 8x + 32$	32	25	20	17	16

Plotting these, we obtain the required graph.

(iii) From the graph we see that the function equals 24 if $x = 1.17$;

$\therefore CQ = 1.17''$ when the area of $APQD$ is 24 sq. in.

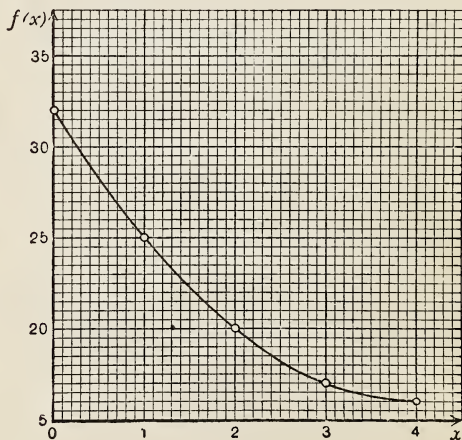


FIG. 8.

(iv) To solve by algebra, we take the equation

$$\begin{aligned}x^2 - 8x + 32 &= 24 ; \\ \therefore x^2 - 8x + 8 &= 0 ; \\ \therefore x &= \frac{8 \pm \sqrt{64 - 32}}{2} = \frac{8 \pm \sqrt{32}}{2} \\ &= \frac{8 \pm 5.657}{2} = \frac{13.657}{2} \text{ or } \frac{2.343}{2} \\ &= 6.828 \text{ or } 1.171.\end{aligned}$$

The value 6.828 is excluded by geometrical considerations ;

$$\therefore x = 1.171.$$

EXERCISE II. c.

[Examples 1 and 2 require the use of geometrical instruments.]

1. AB is the diameter of a circle APB ; PN is the perpendicular from P to AB ; $AB = 10$ cm. By taking different positions for P and making the necessary measurements, construct a table of values connecting the lengths of AP and AN . Represent it graphically, and find the length of AN when $AP = 2AN$.

2. With the notation and data of Ex. 1, make a table of values for the relation between AN and the area of the triangle APN ; construct the graph, and find the length of AN for which the triangle APN is of maximum area.

If $AN = x$ cm., express the area of the triangle APN as a function of x .

3. A funnel is made in the shape of a pyramid of height a inches standing on a square base of side $2x$ inches. The funnel holds 48 cu. inches of liquid. Prove that (i) $a = \frac{36}{x^2}$; (ii) if the area of the four slant faces is y sq. inches, $y = 4x\sqrt{x^2 + a^2}$.

Tabulate the values of y for x equal to 1, 2, 3, 4, 5, and from the graph find the value of x for which y is least.

4. The perimeter of a right-angled triangle is 20 cm., and the shortest side is x cm. ; if the area is A sq. cm., express A as a function of x . Find from a graph the value of x for which the area is a maximum.

5. The base of an open cistern is a square of side x feet ; its volume is 200 cu. ft. The total area of the base and sides is A sq. ft. Express A as a function of x . Represent A graphically for values of x from 4 to 12. Find the depth of the cistern (i) when A is least, (ii) when the total area of the base and sides is 200 sq. ft.

6. A sheet of tin is 24 inches square ; equal squares are cut out at the four corners, and the sides are then turned up to make a rectangular box ; if the side of each square is x inches and the volume of the box is V cu. inches, express V as a function of x , and find graphically the maximum capacity of the box.

7. The base of a box is a square of side x feet, and the length of cord (exclusive of knots) needed for binding it once round each way is 30 feet. Express the volume V cu. ft. of the box as a function of x , and find from a graph the maximum volume of the box.

8. From a circular cone, height 12 cm., base diameter 12 cm., the greatest circular cylinder of radius r cm. is cut. Express its volume V cu. cm. as a function of r , and find from a graph the volume of the greatest cylinder.

9. Fill in the gaps in the following table, and so construct a table of values of the function $4 + 2x - x^2$ for values of x from -2 to $+4$:

x	-2	-1	0	1	2	3	4
4	4					4	
$+ 2x$	-4					6	
$- x^2$	-4					-9	
$4 + 2x - x^2$	-4					1	

Draw accurately the graph of this function, and from your graph answer the following questions :

- (i) For what values of x does $4 + 2x - x^2 = 0$?
- (ii) Between what values of x is $4 + 2x - x^2$ positive ?
- (iii) What is the maximum value attained by the function $4 + 2x - x^2$, and for what value of x does the function have its maximum value ?

- (iv) For what values of x does $4 + 2x - x^2 = 1\frac{1}{2}$?
Solve the equation $2(4 + 2x - x^2) = 3$.
- (v) If $2x - x^2 = -2$, what is the value of $4 + 2x - x^2$?
For what values of x does $4 + 2x - x^2$ have this value? Solve the equation $2x - x^2 = -2$.
- (vi) Solve the equation $4 + 2x - x^2 = -1$ from the graph, and solve the equation $x^2 - 2x - 5 = 0$ by formula. Compare the results.

10. Construct a table of values of the function $\frac{1}{5}(2x^2 + 2x - 3)$ for values of x from -4 to $+4$ as follows :

x	-4	-3	-2	-1	0	1	2	3	4
$2x^2$	32						8		
$2x$	-8						4		
-3	-3						-3		
$2x^2 + 2x - 3$	21						9		
$\frac{1}{5}(2x^2 + 2x - 3)$	4.2						1.8		

Draw accurately the graph of this function, and from your graph answer the following questions :

- (i) For what values of x is $2x^2 + 2x - 3 = 0$?
- (ii) Between what values of x is $2x^2 + 2x - 3$ negative?
- (iii) What is the minimum value attained by the function $\frac{1}{5}(2x^2 + 2x - 3)$, and for what value of x does the function have its minimum value?
- (iv) For what values of x does $\frac{1}{5}(2x^2 + 2x - 3) = 1$?
Solve the equation $2x^2 + 2x - 3 = 5$.
- (v) If $2x^2 + 2x = 7$, what is the value of $2x^2 + 2x - 3$ and of $\frac{1}{5}(2x^2 + 2x - 3)$? For what values of x does $\frac{1}{5}(2x^2 + 2x - 3)$ have this value? Solve the equation $2x^2 + 2x = 7$.
- (vi) Solve the equation $x^2 + x = 1$ from the graph. Check your result by solving the equation by the formula.

11. Construct a table of values of the function

$$\frac{1}{5}(x^3 - 3x^2 - 4x - 6)$$

for values of x from -3 to $+5$, and draw the graph of this function.

From your graph answer the following questions :

- (i) For what values of x does $x^3 - 3x^2 - 4x - 6 = 0$?
- (ii) For what range of values of x is $x^3 - 3x^2 - 4x - 6$ positive ?
- (iii) What is the smallest value of the function
$$\frac{1}{5}(x^3 - 3x^2 - 4x - 6)$$
 for positive values of x ? For what positive value of x does it have the smallest value ?
- (iv) What is the largest value of the function for negative values of x , and for what negative value of x does it have its largest value ?
- (v) Solve the equation $x^3 - 3x^2 - 4x - 6 = 7$ from the graph.
- (vi) Solve the equations : (a) $x^3 - 3x^2 - 4x + 4 = 0$;
(b) $x^3 - 3x^2 - 4x + 19 = 0$;
(c) $x^3 - 3x^2 - 4x - 16 = 0$.
- (vii) Solve the equation $x^3 - 3x^2 - 4x + 12 = 0$ from the graph, and also by factorising
$$x^3 - 3x^2 - 4x + 12.$$

12. Draw a graph of the function $\frac{1}{x^2}$ for values of x from -3 to $+3$.

- (i) With the same axes, and to the same scale, draw the graph of the function $2 - x$. For what values of x are the functions $\frac{1}{x^2}$ and $2 - x$ equal ?
Hence solve the equation $x^3 - 2x^2 + 1 = 0$.
- (ii) Draw the graph of the function $x - 2$ with the same axes, and solve the equation $\frac{1}{x^2} = x - 2$.
- (iii) Draw the graph of the function $2x + 1$, and solve the equation $x^2(2x + 1) = 1$.

13. Draw graphs of the functions $\frac{1}{x}$ and $x^2 - 4$ with the same axes and to the same scale for values of x from -3 to $+3$.
For what values of x are these functions equal ?
Hence solve the equation $x^3 - 4x = 1$.

Example III. For what range of values of x is the function $5 + 7x - 6x^2$ positive? Find also its greatest value.

$$\begin{aligned} \text{(i) } 5 + 7x - 6x^2 &\equiv (1 + 2x)(5 - 3x) \equiv 2(\tfrac{1}{2} + x)(-3)(x - \tfrac{5}{3}) \\ &\equiv -6(x - \tfrac{5}{3})(x + \tfrac{1}{2}). \end{aligned}$$

The function is positive if $(x - \tfrac{5}{3})(x + \tfrac{1}{2})$ is negative.

If $x > \tfrac{5}{3}$, each factor is positive;

$\therefore$ the product is positive.

If $-\tfrac{1}{2} < x < \tfrac{5}{3}$ the first factor is negative, the second is positive;

$\therefore$ the product is negative.

If $x < -\tfrac{1}{2}$ each factor is negative;

$\therefore$ the product is positive;

$\therefore$ the range of values is $-\tfrac{1}{2} < x < \tfrac{5}{3}$.

$$\begin{aligned} \text{(ii) } 5 + 7x - 6x^2 &\equiv 5 - 6\left(-\frac{7x}{6} + x^2\right) \\ &= 5 - 6\left(\frac{7}{12} - x\right)^2 + \frac{49}{24} \\ &= 7\frac{1}{24} - 6\left(\frac{7}{12} - x\right)^2. \end{aligned}$$

Now the least value of $\left(\frac{7}{12} - x\right)^2$ is zero.

$\therefore$ the greatest value of $5 + 7x - 6x^2$ is $7\frac{1}{24}$, and the function has this value when $x = \frac{7}{12}$.

14. For the range $-10 \leq x \leq 10$, draw the graphs of the functions:

$$\begin{array}{ll} \text{(i) } 3(x - 1)(x - 2); & \text{(ii) } 3(x + 1)(x - 2); \\ \text{(iii) } 3(x + 1)(x + 2); & \text{(iv) } -3(x - 1)(x - 2). \end{array}$$

From your graphs, answer the questions corresponding to those in exercise 11 (i), (ii), (iii), above.

15. For the range $-5 \leq x \leq +15$ draw the graphs of the functions

$$\begin{array}{ll} \text{(i)} & 2(x-1)^2 - 3; \\ \text{(iii)} & 4(x-3)^2 + 5; \end{array} \quad \begin{array}{ll} \text{(ii)} & -2(x-1)^2 + 3; \\ \text{(iv)} & -2(x+3)^2 + 5. \end{array}$$

Find the greatest values of the functions (ii) and (iv), the least values of (i) and (iii).

16. (a) Express each of the quadratic functions in Exercise II. a, 1, 2, 3, and II. c, 9, in the "completed square" form, $a(x-h)^2 + k$.

(b) Draw the graph of each function in (a). Find the highest or lowest point of each graph.

(c) Express $ax^2 + bx + c$ in the same form.

17. (i) With the same set of axes and the same scales draw the graphs of $2(x-1)^2 + k$, where $k = -4, -2, 0, 2, 4$.

(ii) As in (i) for $-2(x-1)^2 + k$, where $k = -4, -2, 0, 2, 4$.

(iii) As in (i) for $a(x-1)^2 - 3$, where $a = -1, 0, 1, 2$.

(iv) As in (i) for $2(x-h)^2 - 3$, where $h = -3, 0, +3$.

D. FURTHER WORK IN QUADRATIC EQUATIONS AND FUNCTIONS.

EXERCISE II. d.

1. Solve the equations (the solution in each case includes verification by substitution) :

$$\begin{array}{ll} \text{(a)} & 2(x-1)^2 = 0; \\ \text{(c)} & 2(x-1)^2 + 3 = 0; \\ \text{(e)} & a(x-h)^2 + k = 0; \end{array} \quad \begin{array}{ll} \text{(b)} & 2(x-1)^2 - 3 = 0; \\ \text{(d)} & -2(x-1)^2 + 3 = 0; \\ \text{(f)} & ax^2 + bx + c = 0. \end{array}$$

2. Compute the intersections of the following graphs with the x -axis :

$$\begin{array}{ll} \text{(a)} & y = 2(x-1)^2; \\ \text{(c)} & y = 2(x-1)^2 + 3; \\ \text{(e)} & y = a(x-h)^2 + 3; \end{array} \quad \begin{array}{ll} \text{(b)} & y = 2(x-1)^2 - 3; \\ \text{(d)} & y = -2(x-1)^2 + 3; \\ \text{(f)} & y = ax^2 + bx + c. \end{array}$$

Note.—The graphs in 2 are parabolas.

3. Using the fact that every point on the x -axis represents a "real number" state the real solutions (i.e. the solutions which are "real numbers") of the corresponding equations given in Ex. 17, in II. c.

4. From their graphs find for what "real" values of x the functions given in Ex. 2 are positive.

5. Verify the following identities :

$$ax^2 + bx + c \equiv a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$$

$$\equiv a\left\{x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right\}\left\{x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right\}.$$

6. Express each of the quadratic functions in Exercise II. d, 1, above, in the form $a(x - r)(x - s)$.

(a) Using the results in Ex. 5 above, establish the following theorem :

In the equation $ax^2 + bx + c = 0$,
whenever $b^2 - 4ac > 0$, the roots of the equation are
"real numbers" ;

Whenever $b^2 - 4ac = 0$, the equation has one
"real" root, or, as we prefer to say, the roots of
the equation are "real" and equal :

Whenever $b^2 - 4ac < 0$, the equation has no "real"
roots, but has two "imaginary" roots.

(b) *Definition.*—The expression $b^2 - 4ac$ is accordingly called the *discriminant* of the equation.

(c) Find quadratic equations, $ax^2 + bx + c = 0$, illustrating the theorem in (a). Accompany each with a sketch of the corresponding function $ax^2 + bx + c$.

7. Draw graphs of the functions x^2 and $x - 1$ with the same axes and to the same scale. When are these functions equal ?

What does this tell you about the solutions of the equation $x^2 - x + 1 = 0$?

8. State whether the roots of the following equations are real and distinct, real and coincident, or imaginary. The actual roots are *not* required.

- | | |
|-----------------------------|------------------------------|
| (i) $x^2 - 3x - 5 = 0$; | (ii) $x^2 + 97x - 173 = 0$; |
| (iii) $x^2 - 2x + 5 = 0$; | (iv) $x^2 - 14x + 49 = 0$; |
| (v) $3x^2 + 105x + 1 = 0$; | (vi) $3x^2 + x + 105 = 0$. |

9. Form the equations whose roots are

- (i) $2, -1$; (ii) $0, \frac{3}{4}$; (iii) $a + b, a - b$; (iv) a, b, c ;
(v) $\pm \sqrt{2}$; (vi) $\sqrt{2} + 1, \sqrt{2} - 1$; (vii) $\frac{a}{b}, -\frac{c}{d}$.

10. Express as simply as possible the roots of $ax^2 + 2bx + c = 0$.

11. Find the greatest value of a for which $x^2 - 5x + a = 0$ has real roots.

12. What must be added to $4x^2 - 5x$ to make the result a perfect square? Find the least value of this function.

13. If x is real, find (i) the greatest value of $1 + 2x - x^2$, (ii) the least value of $5x^2 - x$.

14. If x is real, prove that $x^2 - 3x + 3$ is positive.

15. For what range of values of x is (i) $3 - 2x - 8x^2$ positive? (ii) $2x^2 - x - 3$ negative?

16. How is x limited if $x^2 + 3x - 4$ is positive?

17. What is the condition that $ax^2 + ax + c$ is positive for all real values of x ?

18. If x is real, what limits are there to the value of $x + \frac{1}{x}$?

19. (i) Solve the equation $\frac{x}{x^2 + 1} = k$.

(ii) What is the condition that the roots of this equation are real?

(iii) If x is real, what is the least value of $\frac{x}{x^2 + 1}$?

20. Two of the roots of $x^3 + qx + r = 0$ are $4, 3$; what is the other root? Find q and r .

E. FUNCTIONAL NOTATION.

Any expression whose value can be determined when the value of x is known, can be represented by the symbol $f(x)$.

$f(x)$ is shorthand for the words "a function of x ."

In a particular question $f(x)$ might be used to represent the function $5x^4 - 2x + 8$.

Then $f(2)$ would mean the value of this function when $x = 2$.

$$\begin{aligned}\therefore f(2) &\text{ would mean } 5(2)^4 - 2(2) + 8 \\ &= 80 - 4 + 8 \\ &= 84,\end{aligned}$$

$$\begin{aligned}\text{and } f(-1) &\text{ would mean } 5(-1)^4 - 2(-1) + 8 \\ &= 5 + 2 + 8 \\ &= 15,\end{aligned}$$

$$\begin{aligned}\text{and } f(0) &\text{ would mean } 5(0)^4 - 2(0) + 8 \\ &= 8.\end{aligned}$$

Example IV. If $f(x) \equiv x^2 - 3 + \frac{1}{x}$, find the value of $f(5)$ and $f(2a)$.

$$\begin{aligned}f(5) &= 5^2 - 3 + \frac{1}{5} = 25 - 3 + \frac{1}{5} = 22\frac{1}{5}, \\ f(2a) &= (2a)^2 - 3 + \frac{1}{2a} = 4a^2 - 3 + \frac{1}{2a}.\end{aligned}$$

Example V. If $f(x) \equiv x^2 + 2x$, find the value of $f(x+h) - f(x)$.

$$\begin{aligned}\text{Here } f(x+h) &= (x+h)^2 + 2(x+h) \\ &= x^2 + 2xh + h^2 + 2x + 2h; \\ \therefore f(x+h) - f(x) &= x^2 + 2xh + h^2 + 2x + 2h - x^2 - 2x \\ &= 2xh + h^2 + 2h \\ &= h(2x + 2 + h).\end{aligned}$$

EXERCISE II. e.

1. If $f(x) \equiv x^2 + 2$, find the values of $f(1)$; $f(0)$; $f(-1)$; $f(2a)$; $f(b^3)$.

2. If $f(x) \equiv 10^x$, find the values of $f(1)$; $f(2)$; $f(0)$; $f(-1)$; $f(2x)$.

3. If $f(x) \equiv \log x$, find the values of $f(1000)$; $f(2)$; $f(20)$; (1) ; $f(x^3)$.

4. If $f(x) \equiv x^2 - 3x + 5$, find the values of $f(2)$; $f(x + h)$; $f\left(\frac{1}{x}\right)$.

5. If $f(x) \equiv x^2 + 3x$, find the value of $\frac{f(x+h) - f(x)}{h}$.
What is the approximate value of this when h is small compared with x ?

6. If $f(x) \equiv \frac{1}{x}$, find the values of $f(1)$ and $\frac{f(1+h) - f(1)}{h}$.
What is the approximate value of this last expression when h is small?

7. If $f(x) \equiv x^2 + 5$, simplify

(i) $f(3x)$;

(ii) $f(x+1) + f(x-1) - 2f(x)$.

8. If $f(x) \equiv x^2$, simplify

(i) $f(x^3)$;

(ii) $f(x+h) - f(x) - \{f(x) - f(x-h)\}$.

9. If $f(x) \equiv (x+7)(7-x)$, show that $f(-x)$ equals $f(x)$.
Construct another function which has this property.

10. If $f(x) \equiv \frac{1+x^2}{x}$, show that $f\left(\frac{1}{x}\right)$ is equal to $f(x)$.
Construct another function which has this property.

11. For what values of x does $f(x) = 0$ if $f(x)$ is

$$(x-1)(x-2) ?$$

12. If $f(x) \equiv 2x^2 - x$, solve the equation $f(x) = f(x-1)$.

13. If $f(x) \equiv \frac{1}{1-x}$ simplify $f(f(x))$.

Find simple functions which have the following properties :

14. $f(3) = 0, f(4) = 0$. 15. $f(3) = 0, f(4) = 8$.

16. $f(3) = 0, f(x) \rightarrow \infty$ when $x \rightarrow 1$, and also $f(1,000,000)$ is approximately equal to 3.

17. $f(x) = f(-x), f(1) = 3$. 18. $f(x) + f(y) = f(xy)$.

19. $f(x) \times f(y) = f(x + y)$. 20. $f(x) = -f(-x), f(1) = 0$.

21. $f(1) = f(3) = f(5) = -1, f(2) = f(4) = f(6) = +1$.

CHAPTER III.

LIMITS AND GRADIENTS.

It has already been noticed, in connection with the graphical representation of functions, that a function may have no meaning for one or more special values of the variable (e.g., $\frac{36}{x}$ has no meaning when $x = 0$, see p. 24).

We shall now examine more closely the behaviour of a function in the neighbourhood of such values of the variable. Examples I. and II. illustrate the meaning of a "Limit" of a function.

Example I. Plot the values of the functions :

$$(i) \ 1 - \frac{1}{n}, \quad (ii) \ 1 + \frac{1}{n}; \quad (iii) \ 1 + (-1)^n \cdot \frac{1}{n}$$

for positive *integral* values of n .

$$(i) \ 1 - \frac{1}{n}.$$

We have the following table of values :

$n = 1$	2	3	4	5	6	7	8
$1 - \frac{1}{n} = 0$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$

which are represented in Fig. 9.

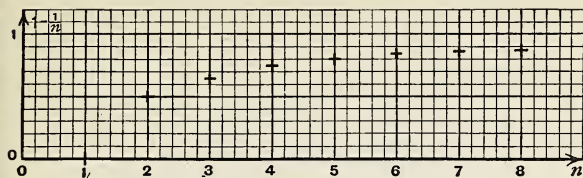


FIG. 9.

$$(ii) \ 1 + \frac{1}{n}.$$

We have the following table of values :

$n = 1$	2	3	4	5	6	7	8
$1 + \frac{1}{n} = 2$	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{6}{5}$	$\frac{7}{6}$	$\frac{8}{7}$	$\frac{9}{8}$

which are represented in Fig. 10.

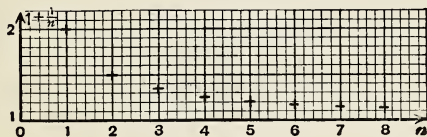


FIG. 10.

$$(iii) \ 1 + (-1)^n \cdot \frac{1}{n}.$$

We have the following table of values :

$n = 1$	2	3	4	5	6	7	8
$1 + (-1)^n \cdot \frac{1}{n} = 0$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{5}{4}$	$\frac{4}{5}$	$\frac{7}{6}$	$\frac{6}{7}$	$\frac{9}{8}$

which are represented in Fig. 11.

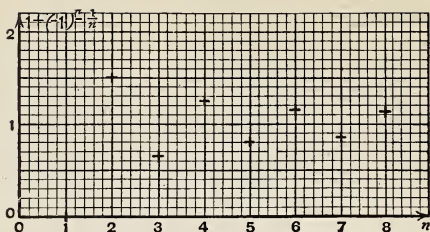


FIG. 11.

Either from the graphs or from examining the functions direct we note that

(i) $1 - \frac{1}{n}$ is always less than 1, and that, as n increases, the function steadily increases and approaches the limit 1.

(ii) $1 + \frac{1}{n}$ is always greater than 1, and that as n increases the function steadily decreases and approaches the limit 1.

(iii) $1 + (-1)^n \cdot \frac{1}{n}$ is alternately less and greater than 1, and that as n increases the function increases and decreases alternately, but approaches the limit 1.

In each case it is possible to find a value of n for which and for all greater values the function differs from 1 by less than a given amount, however small.

E.g., each function differs from 1 by less than 0.001 if $n > 1000$.

But it is impossible to find a value of n for which any of the functions exactly equals 1.

Under these conditions we say that :

As n increases without limit, the limit of $1 + \frac{1}{n}$ is 1

[but the limit is not attained in this case]. And we write it as follows :

$$\text{Lt}_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1, \text{ or } \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1.$$

$$\text{Similarly } \text{Lt}_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = 1 \text{ and } \text{Lt}_{n \rightarrow \infty} \left[1 + (-1)^n \cdot \frac{1}{n}\right] = .$$

Example II. What is the limit of $\frac{(1+h)^2 - 1}{h}$ as $h \rightarrow 0$?

$$\frac{(1+h)^2 - 1}{h} = \frac{1 + 2h + h^2 - 1}{h} = \frac{2h + h^2}{h}.$$

If $h = 0$, $\frac{2h + h^2}{h}$ is $\frac{0}{0}$, which is a meaningless expression.

If $h \neq 0$, $\frac{2h + h^2}{h} = 2 + h$.

The smaller h becomes, the closer $2 + h$ approaches to 2. It never attains the value 2, for h is never exactly 0. But we can choose h so that $2 + h$ differs from 2, for that and all smaller values of h , by less than any given amount, however small ;

$\therefore$ its limit is 2 ;

$$\therefore \text{Lt}_{h \rightarrow 0} \frac{(1+h)^2 - 1}{h} = 2 \text{ (limit not attained).}$$

The following statements should be carefully compared.

$$\text{If } h \neq 0, \frac{2h + h^2}{h} = 2 + h.$$

$$\text{If } h = 0, 2 + h = 2.$$

However, if $h = 0$, $\frac{2h + h^2}{h}$ is meaningless, *i.e.* the division is impossible.

$$\text{If } h \rightarrow 0, 2 + h \rightarrow 2 ;$$

$$\text{and if } h \rightarrow 0, \frac{2h + h^2}{h} \rightarrow 2.$$

EXERCISE III. a.

1. (i) Plot the values of $\frac{2n+1}{n+1}$ for integral values of n from 3 to 10.
 (ii) Can you find a value of n for which $\frac{2n+1}{n+1}$ differs from 2 by less than 0.001?
 (iii) Can you find a value of n for which $\frac{2n+1}{n+1}$ equals 2?
 (iv) What is the limit of $\frac{2n+1}{n+1}$ as n increases without limit?
2. (i) If $s_1 = 1$; $s_2 = 1 + \frac{1}{2}$; $s_3 = 1 + \frac{1}{2} + \frac{1}{4}$; $s_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, etc., plot the values of s_n for values of n from 1 to 5.
 (ii) Can you find a value of n for which s_n differs from 2 by less than 0.001?
 (iii) Can you find a value of n for which s_n equals 2?
 (iv) What is $\text{Lt}_{n \rightarrow \infty} s_n$?
3. (i) Find the values of $\frac{(0.1)^n}{1 + (0.1)^n}$ for integral values of n from 1 to 4.
 (ii) What is $\text{Lt}_{n \rightarrow \infty} \frac{(0.1)^n}{1 + (0.1)^n}$?
4. (i) Plot the values of $\frac{2^n}{1 + 2^n}$ for integral values of n from 1 to 4.
 (ii) What is $\text{Lt}_{n \rightarrow \infty} \frac{2^n}{1 + 2^n}$?
 (iii) For what integral value of n does $\frac{2^n}{1 + 2^n}$ differ from 1 by less than $\frac{1}{100}$?
5. (i) Find the values of $\frac{x^2 - 1}{x^2 - x}$ for $x = 2, 1.5, 1.1, 1.01$.
 (ii) What is $\text{Lt}_{x \rightarrow 1} \frac{x^2 - 1}{x^2 - x}$?
 (iii) Is this limit attained?
 (iv) Find a value of x for which $\frac{x^2 - 1}{x^2 - x}$ differs from 2 by less than 0.001.

6. Fig. 12 shows part of the graph of $y = \frac{1}{2}x^3$.

$$ON = 1, NM = h.$$

NP , MQ , are ordinates and PR is perpendicular to MQ .

- (i) Express $\frac{RQ}{PR}$ in terms of h .
- (ii) What is the limit of $\frac{RQ}{PR}$ as $h \rightarrow 0$? Interpret this result geometrically.

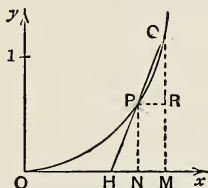


FIG. 12.

7. If a stone is dropped in a vacuum, it falls s feet in t seconds, where $s = 16t^2$; the graph of this is represented in Fig. 13.

$$ON = \frac{1}{2}, NM = h.$$

NP , MQ , are ordinates and PR is perpendicular to MQ .

- (i) Express $\frac{RQ}{PR}$ in terms of h .
- (ii) What is the limit of $\frac{RQ}{PR}$ as $h \rightarrow 0$? Interpret this result.

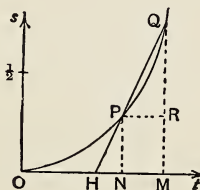


FIG. 13.

8. (i) Draw the graph of $5x - x^2$ from $x = 0$ to 5.
 (ii) Draw the ordinate corresponding to $x = 1.5$. What is its length?
 (iii) Take any point A on the x -axis and call $OA = a$.
 Give a geometrical meaning to the ratio

$$\frac{[5(a+h) - (a+h)^2] - [5a - a^2]}{h}.$$

- (iv) What is the limit of this function when $h \rightarrow 0$?
 (v) For what value of a is this limit equal to 0? Interpret this result geometrically.

9. (i) Draw the graph of $\frac{1}{x}$ from $x = 4$ to $\frac{1}{4}$.
 (ii) Give a geometrical meaning to the ratio $\frac{\frac{1}{2+h} - \frac{1}{2}}{h}$.
 (iii) What is the limit of this ratio when $h \rightarrow 0$? Interpret this result.
 (iv) Give a geometrical meaning to the ratio $\frac{\frac{1}{a+h} - \frac{1}{a}}{h}$; evaluate its limit when $h \rightarrow 0$; and find the value of a for which this limit equals $-\frac{1}{4}$.

CALCULATION OF RATES OF CHANGE FROM STATISTICS

Example III. The following table gives the height of the mercury barometer at intervals of two hours during a day :

Time	8 a.m.	10 a.m.	Noon	2 p.m.	4 p.m.	6 p.m.	8 p.m.
Height in inches	28.57	28.65	28.85	29.10	29.22	29.12	29.0

Find the average rate at which the barometer was rising :

- (i) between 10 a.m. and noon ;
 (ii) between noon and 2 p.m. ;
 (iii) between 4 p.m. and 6 p.m.

Draw the barograph for the day, and find the rate at which the barometer was probably rising at noon.

- (i) Between 10 a.m. and noon the barometer rose $0.2''$ in 2 hours ;
 $\therefore$ the average rate of rise was $0.1''$ per hour.
- (ii) Between noon and 2 p.m. the barometer rose $0.25''$ in 2 hours ;
 $\therefore$ the average rate of rise was $0.125''$ per hour.
- (iii) Between 4 p.m. and 6 p.m. the barometer fell $0.1''$ in 2 hours ;
 $\therefore$ the average rate of rise was $-0.05''$ per hour.

Fig. 14 gives the required barograph.

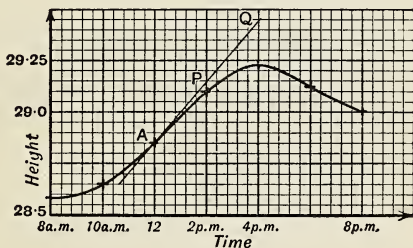


FIG. 14.

It is a curve and not a straight line, because the rate is altering throughout the day. If it continued to rise at the same rate after 12 o'clock as it is rising at 12 o'clock, the barograph would be the straight line formed by drawing the tangent APQ at the point A , which corresponds to 12 o'clock on the graph. In the figure, this tangent is drawn by eye, and cuts the 2 o'clock and 4 o'clock lines (or any other convenient lines) at P and Q . Now the difference in heights registered by P , Q is $0.30''$.

$\therefore$ the rate of rise at A is $0.30''$ in 2 hours,
or $0.15''$ per hour.

EXERCISE III. b.

1. The following table gives the population of the United Kingdom for various years :

Year - -	1840	1850	1860	1870	1880	1890	1900
Population (in millions) -	27·0	27·7	29·3	31·8	35·2	38·1	42·0

[Give answers in thousands per year.]

- What was the average rate of increase of population in the period 1840 to 1850 ?
- What was the average rate of increase of population for the whole period 1840 to 1900 ?
- During what period of ten years was the population increasing most rapidly, and what was the rate per year then ?

2. At the end of a minutes a car has travelled x miles, and at the end of b minutes it has travelled y miles. What was its average speed

- for the first a minutes ?
- for the first b minutes ?
- during the period a minutes to b minutes ?

If in the above question $a = 3$ and $b = 4$, what was the average speed of the car during the 4th minute ?

3. The following table gives the distances a car travels, starting from rest :

Time in minutes -	5	10	15	20	25	30	35	40	45
Distance in miles -	0·7	2·6	5·2	8·3	11·4	14·2	16·7	19	20·5

[Give answers in the form miles per hour.]

- What was the average speed of the car for the first 5 minutes ?
- What was the average speed for the first 20 minutes ?

- (c) What was the average speed for the whole period, 45 minutes ?
- (d) Draw a graph to illustrate the motion of the car, and by drawing a tangent to the graph at the point determined by $t = 35$, find the probable speed of the car 35 minutes after it started.
- (e) By drawing a tangent to the graph where it appears to be steepest, find the greatest speed attained by the car.

4. An elevator ascends 90 ft. in 40 seconds, and its height h feet at intervals of 5 seconds is given by the following table :

t	5	10	15	20	25	30	35	40
h	4	15	30	45	62	77	86	90

Draw a graph to illustrate the motion, and draw tangents to the graph at the points determined by $t = 10, t = 20, t = 30$.

- (i) What is the average speed of the elevator for the first 10 seconds ?
- (ii) Estimate from your graph the speed of the elevator after 10 seconds ?
- (iii) What is the average speed of the elevator for the first 20 seconds ?
- (iv) Estimate its speed after 20 seconds.
- (v) What is the average speed of the elevator for the last 10 seconds ?
- (vi) Estimate its speed after 30 seconds.

5. Water runs out of a bath, and the volume that runs out is given by the following table :

Time in seconds	-	-	5	10	15	20	25
Volume in cubic feet	-	6	9	10.5	11.2	11.6	

Draw a graph and find the rate of flow at the end of each period of 5 seconds. Show from your results that the rate of flow is proportional to the volume of water remaining in the bath, there being 12 cubic feet of water in the bath originally.

SUMMARY OF RESULTS.

(i) It is possible to have a function of x to which, for one or more special values of x , no meaning can be given.

E.g. $\frac{x^2 - 1}{x - 1}$ has no meaning when $x = 1$.

(ii) In such cases, the function may tend towards a definite limit as x tends to that value ; but the limit is not attained.

E.g. $\text{Lt}_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$, but no value of x exists for which $\frac{x^2 - 1}{x - 1}$ equals 2. Compare Example II. above.

(iii) If a function of x is represented by a graph, the rate at which the function is increasing for any value of x is represented by the slope of the tangent to the graph at the point corresponding to that value of x , provided that the units used in drawing the graph are taken into account.

CALCULATION OF RATES OF CHANGE FROM FORMULAE.

Example IV. Draw the graph of $y = 2 + 3x + x^2$, and find the change of y per unit increase of x , when (i) $x = 2$, (ii) $x = a$.

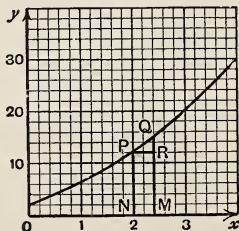


FIG. 15.

We have, by calculation, the table :

$x = 0$	1	2	3	4
$y = 2$	6	12	20	30

which is represented in Fig. 15.

(i) When $ON = 2$, $NP = 2 + 3 \times 2 + 2^2 = 2 + 6 + 4 = 12$.

When $OM = 2 + h$, $MQ = 2 + 3(2 + h) + (2 + h)^2 = 12 + 7h + h^2$;

$\therefore$ when x increases by h , y increases by $MQ - NP$ or $RQ = 7h + h^2$;

$\therefore$ the average increase of y per unit increase of x

$$= \frac{7h + h^2}{h} = 7 + h$$

or

$$\frac{RQ}{PR} = 7 + h;$$

$$\therefore \text{Lt}_{h \rightarrow 0} \frac{RQ}{PR} = 7;$$

$\therefore$ at $x = 2$, the rate of increase of y with respect to x is 7.

(ii) When $ON = a$, $NP = 2 + 3a + a^2$.

When $OM = a + h$,

$$MQ = 2 + 3(a + h) + (a + h)^2 = 2 + 3a + 3h + a^2 + 2ah + h^2;$$

$$\therefore RQ = MQ - NP = 3h + 2ah + h^2;$$

$$\therefore \frac{RQ}{PR} = \frac{3h + 2ah + h^2}{h} = 3 + 2a + h;$$

$\therefore$ at $x = a$, the rate of increase of y with respect to x ,

$$= \text{Lt}_{h \rightarrow 0} (3 + 2a + h)$$

$$= 3 + 2a.$$

Definition. In Fig. 15, $\frac{RQ}{PR}$ is called the *average gradient* of the graph over the interval NM .

The limit of $\frac{RQ}{PR}$ when h tends to 0, or $\text{Lt}_{h \rightarrow 0} \frac{RQ}{PR}$ is called the *gradient* of the graph at P .

Fig. 16 illustrates the way in which the "average gradient" changes as the point Q is taken nearer and nearer to P .

Q_1, Q_2, Q_3 are successive positions of Q , while P remains fixed.

The average gradient over the interval NM is $\frac{RQ}{PR}$.

The limiting position to which the secant PQ tends as $PQ \rightarrow 0$ is indicated by the *tangent* PPT' .

$\text{Lt}_{h \rightarrow 0} \frac{RQ}{PR} = \text{gradient of the graph at } P = \text{gradient of the tangent } PT = \frac{NP}{TN}$.

The rate of change of y with respect to x , at $x = a$, is given by the gradient of the curve at P , i.e. by the gradient of the tangent to the curve at P .

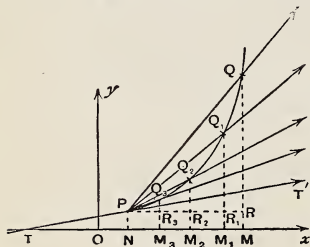


FIG. 16.

NOTE.—By "RQ" above is meant the number of vertical units in the line-segment RQ, and by "PR" is meant the number of horizontal units in the line-segment PR. In Figs. 12 and 13 these units are equal; in Fig. 15 these units are *not* equal.

EXERCISE III. c.

1. What is the gradient of each line shown in Figs. 17-19?

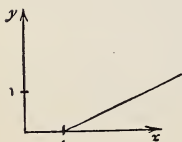


FIG. 17.

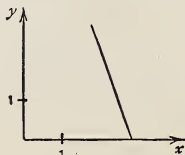


FIG. 18.

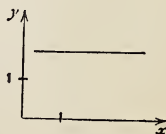


FIG. 19.

2. Fig. 20 represents a hill; horizontal scale x -axis is 1 inch : 100 yards; vertical scale y -axis is 1 inch : 10 feet. Find (i) the average gradient from A to B , (ii) the average gradient from P to Q , (iii) the gradients at C , D , Q .

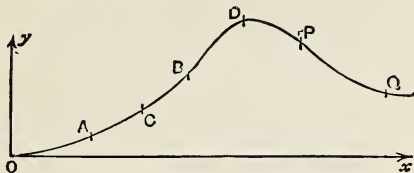


FIG. 20.

3. Draw lines of gradients $\frac{3}{4}$, $-\frac{1}{2}$, $-\frac{4}{3}$, $+2$, -3 , $\frac{1}{3}$, k , $-\frac{1}{k}$.

4. Find the co-ordinates of a point on the curve in Fig. 20, for the scale given in Ex. 2, where the slope is $\frac{1}{40}$.

5. (i) Draw the graph of $y = 2x + 3$.
 (ii) Show that the points (1, 5) and (4, 11) lie on it.
 (iii) What is the average gradient of the graph between these two points?
 (iv) What is the average gradient of the graph over the interval $x = 2$ to $x = 6$?
 (v) What is the value of y when $x = a$ and when $x = a + h$, and what is the average gradient of the graph over this interval?
 (vi) Why does the average gradient of the graph not depend on the values of either a or h ?
6. (i) Draw the graph of $y = 7 - 5x$.
 (ii) What is the average gradient of the graph over the interval $x = 3$ to $x = 7$?
 (iii) What is the gradient of the graph when $x = a$?

7. What is the gradient of the straight line joining the points (2, 5) and (7, 8), (x_1, y_1) and (x_2, y_2) , $(a, 0)$ and $(0, b)$?

8. A marble rolling down an inclined plane travels s feet in t seconds, where $s = 3t^2$.

- (i) How far has the marble travelled in 1 second ?
What is its average speed for the 1st second ?
- (ii) How far has the marble travelled in 2 seconds ?
What is its average speed for the 1st 2 seconds ?
What is its average speed during the 2nd second ?
- (iii) How far has the marble travelled in 2.1 seconds ?
What is its average speed in the interval 2 seconds to 2.1 seconds ?
- (iv) How far has it travelled in $(2 + h)$ seconds ? What is its average speed in the h seconds between 2 seconds and $(2 + h)$ seconds ?
- (v) What does your answer to Question (iv) become when $h = 0.1$ second, when $h = 0.01$ second, and when $h = 0.000001$ second ?

What is its speed exactly 2 seconds after it begins to move ?

Draw a graph to illustrate the motion from $t = 0$ to $t = 3$, and find its speed after 2 seconds by drawing a tangent.

9. The distance d ft. that a stone has fallen after t seconds is given by the formula $d = 16t^2$.

- (i) How far has the stone fallen after 3 seconds ? What is its average speed for the first 3 seconds ?
- (ii) How far has the stone fallen after 2 seconds ? What is its average speed during the third second ?
- (iii) How far has the stone fallen after 2.9 seconds ?
What is its average speed during the interval from 2.9 seconds to 3 seconds ?
- (iv) How far does it fall in $(3 - h)$ seconds ? What is its average speed during the h seconds from $(3 - h)$ seconds to 3 seconds ?
- (v) What does your answer to Question (iv) become when $h = 0.1$ second, when $h = 0.01$ second, and when $h = 0.000001$ second ?

What is the velocity of the stone 3 seconds after it is dropped ?

10. In the same way as in Ex. 9 work out the speed of the stone 1 second after it is dropped.

11. In the same way as in Ex. 9 work out the speed of the stone a seconds after it is dropped. Evaluate your result for $a = 1, 2$ and 3 , and compare with previous results.

12. Draw a line AB of length 10 cm. and describe a semicircle with AB as diameter; P is any point on the semicircle and PN is the perpendicular to AB . Let $AN = x$ and $NP = y$. Find by measurement or calculation the average gradient of the semicircle over the intervals (i) $x = 0$ to 1 ; (ii) $x = 0$ to 2 ; (iii) $x = 3$ to 7 ; (iv) $x = 8$ to 10 . Interpret your results geometrically.

13. If $y = 3x + x^2$, find (i) the average rate of change of y over the interval $x = 1$ to 1.1 ; (ii) the gradient at $x = 1$; (iii) the value of x when the gradient is zero.

14. Calculate the gradient of the graph of $\frac{1}{x}$ when (i) $x = 2$, (ii) $x = a$.

15. Calculate the gradient of the graph of $2x^3$ when $x = c$.

16. A marble rolling down a groove travels s feet in t seconds, where $s = \frac{1}{4}t^2$; find (i) the average rate of change of s over the interval $t = 1$ to $t = 2$, (ii) the rate of change of s when $t = 1$. What does this mean?

17. Show that the gradient of any radius of the circle $x^2 + y^2 = a^2$ is y/x , and that the gradient of the tangent perpendicular to the radius is $-x/y$. What is the gradient of this circle at the point whose $x = 0$? (Compare Ex. 12.)

It is assumed here, of course, that the x - and y -units are equal.

18. The graph of $y = mx + b$ is a straight line, m and b being any constant numbers. Find its gradient.

19. Find the gradient of $y = ax^2 + bx + c$ when $x = 2$.

20. Find the gradient of $y = ax^2 + bx + c$ when $x = x_1$. For what value of x is this gradient equal to zero? What is the geometrical significance of a zero gradient?

21. Give geometrical meanings to the following, taking x to be the x -coordinate of a graph (do not simplify the expressions) :

- (i) $\frac{(x+h)^2 - x^2}{h}$; (ii) $\lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$;
 (iii) $\frac{\sqrt{x+h} - \sqrt{x}}{h}$; (iv) $\lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$;
 (v) $\lim_{h \rightarrow 0} \frac{\pi(r+h)^2 - \pi r^2}{h}$; (vi) $\frac{\frac{4}{3}\pi(r+h)^3 - \frac{4}{3}\pi r^3}{h}$.

22. If $f(x) \equiv x^2$, evaluate $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ and interpret the result.

23. If $f(x) \equiv 5$, what is the gradient of the graph of $f(x)$?

24. Draw the curve $y = 3x + x^2$ (compare Ex. 14 above), the secant through (1, 4) and (3, 18), and the tangent to the curve at the point (1, 4). Compare :

- (i) The average gradient of the curve over the interval $x = 1$ to $x = 1.2$.
 (ii) The gradient of the curve at (1, 4).
 (iii) The gradient of the tangent. (Compare Exs. 3-7, 19.)

SUMMARY OF RESULTS.

The Linear Function.

The gradient of the straight line through the points (x_1, y_1) and (x_2, y_2) is

$$\frac{y_2 - y_1}{x_2 - x_1}$$

The equation of the straight line through $(0, b)$ having gradient m is

$$y = mx + b.$$

The equation of the straight line through (x_1, y_1) having gradient m is

$$y - y_1 = m(x - x_1).$$

The average gradient of the straight line $y = mx + b$, over any interval, is m : its gradient at any point is m .

Functions in General.

Draw the graph of any function $y = f(x)$: see Fig. 21.

Suppose $ON = x$ and $OM = x + h$, so that

$$PR = NM = h.$$

Then $NP = f(x)$ and $MQ = f(x + h)$.

$$\therefore RQ = f(x + h) - f(x).$$

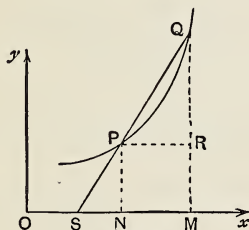


FIG. 21.

The average gradient of the graph over the interval NM is

$$\frac{f(x + h) - f(x)}{h}.$$

And the gradient of the graph at P is

$$\text{Lt}_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}.$$

The gradient of the graph at P is the gradient of the tangent at P to the curve.

CHAPTER IV.

DIFFERENTIATION.

NOTATION.

FIG. 22 represents the graph of any function $y = f(x)$.

If $ON = x$, then $NP = y = f(x)$.

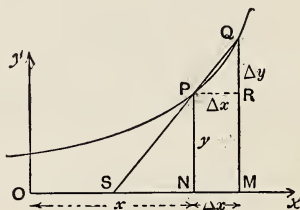


FIG. 22.

If the change in the variable x is NM or Δx (or δx), then the new value of the variable is OM or $x + \Delta x$, the new value of the function is MQ or $f(x + \Delta x)$, and the consequent change in y is $RQ = MQ - NP$ or Δy . Thus

$$ON = x ; \quad OM = x + \Delta x ;$$

$$NP = y = f(x) ; \quad MQ = y + \Delta y = f(x + \Delta x) ;$$

$$\therefore \Delta y = f(x + \Delta x) - f(x),$$

and

$$\frac{RQ}{PR} = \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Now $\frac{RQ}{PR}$ = the gradient of the secant PQ .

$\therefore$ the gradient of the tangent at P

$$\begin{aligned} &= \text{Lt}_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \text{Lt}_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \text{rate of change of } f(x) \text{ with respect to } x. \end{aligned}$$

The expression $\text{Lt}_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$ is written $\frac{dy}{dx}$ or $\frac{d}{dx}y$, and is called the *differential coefficient* of y with respect to x .

The expression $\text{Lt}_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ is written $\frac{df(x)}{dx}$ or $\frac{d}{dx}f(x)$, and is called the *differential coefficient* of $f(x)$ with respect to x , or the “derivative” of $f(x)$, or the “derived function” of $f(x)$.

The process of finding this limit is called “*differentiating with respect to* x .”

SUMMARY.

- (i) $\frac{d}{dx}f(x) = \text{Lt}_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$.
- (ii) If $y = f(x)$, $\frac{dy}{dx} = \text{Lt}_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \text{Lt}_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$.

Example I. Differentiate $3x^2 + 7$ with respect to x .

$$\begin{aligned} \frac{d}{dx}(3x^2 + 7) &= \text{Lt}_{\Delta x \rightarrow 0} \frac{[3(x + \Delta x)^2 + 7] - [3x^2 + 7]}{\Delta x} \\ &= \text{Lt}_{\Delta x \rightarrow 0} \frac{[3\{x^2 + 2x\Delta x + (\Delta x)^2\} + 7] - 3x^2 - 7}{\Delta x} \\ &= \text{Lt}_{\Delta x \rightarrow 0} \frac{6x\Delta x + 3(\Delta x)^2}{\Delta x} \\ &= \text{Lt}_{\Delta x \rightarrow 0} [6x + 3\Delta x], \text{ provided } \Delta x \neq 0, \\ &= 6x. \end{aligned}$$

Example II. Find $\frac{d}{dx}\left(\frac{1}{x}\right)$.

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{x}\right) &= \text{Lt}_{\Delta x \rightarrow 0} \left(\frac{\frac{1}{x + \Delta x} - \frac{1}{x}}{\Delta x} \right) \\ &= \text{Lt}_{\Delta x \rightarrow 0} \frac{x - (x + \Delta x)}{x(x + \Delta x) \cdot \Delta x} \\ &= \text{Lt}_{\Delta x \rightarrow 0} - \frac{\Delta x}{(x^2 + x \cdot \Delta x) \cdot \Delta x} \\ &= \text{Lt}_{\Delta x \rightarrow 0} - \frac{1}{x^2 + x \cdot \Delta x}, \text{ provided } \Delta x \neq 0, \\ &= -\frac{1}{x^2}.\end{aligned}$$

EXERCISE IV. a.

1. (i) Simplify $5(x + \Delta x) - 5x$. (ii) Find $\frac{d}{dx}(5x)$.
2. (i) Simplify $(x + \Delta x)^2 - x^2$. (ii) Find $\frac{d}{dx}(x^2)$.
3. (i) Simplify $[5(x + \Delta x)^2 - 3(x + \Delta x) + 7] - [5x^2 - 3x + 7]$.
(ii) Find $\frac{d}{dx}[5x^2 - 3x + 7]$.
4. (i) If $f(x) \equiv (1 + x)^2$, what is $f(x + \Delta x)$?
(ii) Find $\frac{d}{dx}(1 + x)^2$.
5. Differentiate $x(1 + x)$.
6. Differentiate (i) x^2 ; (ii) $3x^2$; (iii) $7x^2$, and write down the value of $\frac{d}{dx}(100x^2)$.
7. Differentiate (i) x^3 ; (ii) $4x^3$; (iii) $5x^3$; and write down the value of $\frac{d}{dx}(29x^3)$.
8. Given that $\text{Lt}_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^4 - x^4}{\Delta x} = 4x^3$, write down the values of
(i) $\frac{d}{dx}(3x^4)$; (ii) $\frac{d}{dx}(10x^4)$.

9. Given that $\frac{d}{dx} x^5 = 5x^4$, write down the values of

$$(i) \frac{d}{dx}(7x^5); \quad (ii) \frac{d}{dx}(x^5 + 7).$$

10. Given that $\frac{d}{dx} x^6 = 6x^5$, write down the values of

$$(i) \frac{d}{dx}(\frac{3}{4}x^6); \quad (ii) \frac{d}{dx}(5 - x^6); \quad (iii) \frac{d}{dx}(x - 2x^6).$$

11. Express in the limit form the fact that $\frac{d}{dx} x^n = nx^{n-1}$.

12. Use the facts that

$$\text{Lt}_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} = 2x \quad \text{and} \quad \text{Lt}_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} = 3x^2$$

to write down the values of

$$(i) \text{Lt}_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^3 + 2(x + \Delta x)^2] - [x^3 + 2x^2]}{\Delta x};$$

$$(ii) \frac{d}{dx}(3x^2 + x^3); \quad (iii) \frac{d}{dx}(\frac{2}{3}x^3 + \frac{1}{2}x^2 + 5).$$

13. Express as limits

$$(i) \frac{d}{dx} f(x); \quad (ii) \frac{d}{dx} g(x); \quad (iii) \frac{d}{dx}[3f(x) - 5g(x)].$$

If $\frac{d}{dx} f(x) = u$ and $\frac{d}{dx} g(x) = v$, what is $\frac{d}{dx}[3f(x) + 5g(x)]$?

14. Find

$$(i) \frac{d}{dx}(3x); \quad (ii) \frac{d}{dx}(5x); \quad (iii) \frac{d}{dx}(3x \times 5x); \quad (iv) \frac{d}{dx}(3x + 5x).$$

Is $\frac{d}{dx}(3x \times 5x)$ equal to $\frac{d}{dx}(3x) \times \frac{d}{dx}(5x)$?

Is $\frac{d}{dx}(3x + 5x)$ equal to $\frac{d}{dx}(3x) + \frac{d}{dx}(5x)$?

15. (i) Express as limits $\frac{d}{dx}(3x^2)$; $\frac{d}{dx}(5x^3)$; $\frac{d}{dx}(3x^2 + 5x^3)$;
 $\frac{d}{dx}(3x^2 - 5x^3)$; $\frac{d}{dx}(3x^2 \times 5x^3)$; $\frac{d}{dx}(5x^3 \div 3x^2)$.

(ii) Is $\frac{d}{dx}(3x^2) = 3\frac{d}{dx}(x^2)$?

(iii) Is $\frac{d}{dx}(3x^2 \pm 5x^2) = 3\frac{d}{dx}(x^2) \pm 5\frac{d}{dx}(x^2)$?

(iv) Is $\frac{d}{dx}(3x^2 \times 5x^3) = 3\frac{d}{dx}(x^2) \times 5\frac{d}{dx}(x^3)$?

(v) Is $\frac{d}{dx}(5x^3 \div 3x^2) = 5\frac{d}{dx}(x^3) \div 3\frac{d}{dx}(x^2)$?

16. What general formula covers the following facts ?

$$\frac{d}{dx}x^2 = 2x; \quad \frac{d}{dx}x^3 = 3x^2; \quad \frac{d}{dx}x^4 = 4x^3; \quad \frac{d}{dx}x^5 = 5x^4.$$

17. What general formula covers the following facts ?

$$\frac{d}{dx}(7x^2) = 7\frac{d}{dx}(x^2); \quad \frac{d}{dx}(11x^4) = 11\frac{d}{dx}(x^4); \quad \frac{d}{dx}\left(\frac{93}{x}\right) = 93\frac{d}{dx}\left(\frac{1}{x}\right).$$

18. What general formula covers the following facts ?

$$\frac{d}{dx}(x^2 + x^3) = \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3); \quad \frac{d}{dx}(7x^2 - 11x^5) = \frac{d}{dx}(7x^2) - \frac{d}{dx}(11x^5).$$

19. Take simple functions for $f(x)$ and $g(x)$ to show that

$$\frac{d}{dx}[f(x) \times g(x)] \text{ is not equal to } \frac{d}{dx}f(x) \times \frac{d}{dx}g(x).$$

20. Write down special cases of the general formula

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

when n equals (i) 8; (ii) 50; (iii) -1 ; (iv) -3 ; (v) 1; (vi) 0;
 (vii) $\frac{3}{2}$; (viii) $\frac{1}{2}$; (ix) $-\frac{1}{2}$; (x) $-m$.

21. If $y = x^2$, find Δy when $x = 2$, $\Delta x = 0.1$.

22. If $y = \frac{1}{x}$, find Δy when $x = 3$, $\Delta x = \frac{1}{3}$.

23. If $y = x^3$, find Δy when (i) $x = 1$, $\Delta x = 0.1$;
 (ii) $x = 1$, $\Delta x = 0.01$.

24. If $y = 3x^2$, find Δy when $x = 2$, $\Delta x = 0.1$.

25. If $y = x^2 + x$, find Δy when $x = 10$, $\Delta x = -1$.

SUMMARY OF RESULTS.

- (i) $\frac{d}{dx}(x^n) = nx^{n-1}$, where n is integral or fractional, positive or negative.
- (ii) $\frac{d}{dx}(C) = 0$, if C is any constant, i.e. a number independent of x .
- (iii) $\frac{d}{dx}(Cx^n) = C \frac{d}{dx}(x^n)$
 $= Cnx^{n-1}$, where C is any constant.
- (iv) $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$.

Example III. Find $\frac{d}{dx}(x)$; $\frac{d}{dx}\left(\frac{1}{x^2}\right)$; $\frac{d}{dx}(\sqrt{x})$.

Since $\frac{d}{dx}(x^n) = nx^{n-1}$

(i) Put $n = 1$, $\therefore \frac{d}{dx}x = 1x^{1-1} = x^0 = 1$,

or, more simply, $\frac{d}{dx}(x) = \text{Lt}_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{\Delta x} = \text{Lt}_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} = 1$.

(ii) Put $n = -2$, $\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx}(x^{-2}) = -2x^{-2-1} = -2x^{-3} = -2 \times x^{-3}$
 $= -2 \times \frac{1}{x^3} = -\frac{2}{x^3}$.

(iii) Put $n = \frac{1}{2}$, $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$
 $= \frac{1}{2} \times \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2} \times \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$.

Example IV. Find $\frac{d}{dx}\left(3x^4 - 7x + 5 - \frac{2}{x}\right)$.

The expression $= \frac{d}{dx}(3x^4) - \frac{d}{dx}(7x) + \frac{d}{dx}(5) - \frac{d}{dx}\left(\frac{2}{x}\right)$
 $= 3 \frac{d}{dx}(x^4) - 7 \frac{d}{dx}(x) + 0 - 2 \frac{d}{dx}(x^{-1})$
 $= 3(4x^3) - 7(1) + 0 - 2(-1)(x^{-1-1})$
 $= 12x^3 - 7 + 2x^{-2}$
 $= 12x^3 - 7 + \frac{2}{x^2}$.

After a little practice, most of the intermediate steps in the working can be omitted.

SUCCESSIVE DIFFERENTIATION.

$$\text{If } y = x^4, \quad \frac{dy}{dx} = 4x^3;$$

$$\therefore \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (4x^3) = 4 \frac{d}{dx} (x^3) = 4 \times 3x^2 = 12x^2.$$

For the sake of brevity, $\frac{d}{dx} \left(\frac{dy}{dx} \right)$ is written $\frac{d^2y}{dx^2}$: this symbol is called the second differential coefficient of y with respect to x .

$$\text{Similarly } \frac{d^2s}{dt^2} \text{ is short for } \frac{d}{dt} \left(\frac{ds}{dt} \right).$$

$$\text{Example V. Find } \frac{d^2}{dx^2} (3x^2 + 4x - 7).$$

$$\frac{d}{dx} (3x^2 + 4x - 7) = \frac{d}{dx} (3x^2) + \frac{d}{dx} (4x) - \frac{d}{dx} (7) = 6x + 4;$$

$$\therefore \frac{d^2}{dx^2} (3x^2 + 4x - 7) = \frac{d}{dx} (6x + 4) = 6.$$

EXERCISE IV. b.

Differentiate with respect to x the expressions in Examples 1-30.

$$1. x^7. \quad 2. 10x^3. \quad 3. x + \frac{1}{x}. \quad 4. \frac{1}{x^5}.$$

$$5. \frac{3}{x}. \quad 6. \frac{2}{x^2}. \quad 7. 3x^2 - 2x. \quad 8. \frac{1}{4}x^4 - 2.$$

$$9. x^2 \times x^3. \quad 10. (5x)^3. \quad 11. (1 - x)^2. \quad 12. \frac{1}{x^n}.$$

$$13. ax. \quad 14. x^a. \quad 15. \frac{c}{x^3}. \quad 16. b\sqrt{x}.$$

$$17. \sqrt{x^3}. \quad 18. \frac{1}{\sqrt{x}}. \quad 19. x^0. \quad 20. \sqrt[3]{x}.$$

$$21. \frac{2}{5}x^3 - \frac{x}{5}. \quad 22. \frac{x}{10} + \frac{10}{x}. \quad 23. 7x^3 - 2x - 5. \quad 24. \frac{4}{3}\pi x^3.$$

25. $3x^4 - \frac{1}{2}x + 7 - \frac{6}{x}$.

26. $5x^4 + 3x^2 + \frac{1}{x} - \frac{1}{x^2}$.

27. $(x^2 + 1)(x + 2)$. 28. $\left(x + \frac{1}{x}\right)^2$. 29. x^{2n} . 30. x^{-k} .

31. If $y = 3x^2 + 5$, find $\frac{dy}{dx}$ when $x = 1$.

32. If $y = 1 + 2x - x^2$, find $\frac{dy}{dx}$ when $x = -2$.

33. If $y = 2x^3 - 9x^2 + 12x$, find the values of x for which $\frac{dy}{dx} = 0$.

34. If $y = 7x^3 - 9x$, find $\frac{d^2y}{dx^2}$. 35. If $y^2 = x$, find $\frac{d^2y}{dx^2}$.

36. If $y = 6x^2$, prove that $x \frac{dy}{dx} = 2y$.

37. If $y = x(1 - x)$, prove that $1 + \frac{dy}{dx} = \frac{2y}{x}$.

38. If $y = x^3$, prove that $\frac{dy}{dx} \cdot \frac{d^2y}{dx^2} = 18y$.

39. If $s = 100t - 16t^2$, find $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$ when $t = 1$ and $t = 0$.

40. If $y = 7x^4$, express (i) $2y \cdot \frac{dy}{dx}$; (ii) $\frac{d}{dx}(y^2)$ in terms of x .

41. If $y = 2z^2$, $z = 3 + 5x$, express in terms of x , (i) $\frac{dy}{dz}$; (ii) $\frac{dz}{dx}$; (iii) y ; (iv) $\frac{dy}{dx}$; hence show that $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$.

42. If $y = z + z^3$, $z = 1 + x^2$, express in terms of x , (i) $\frac{dy}{dz}$; (ii) $\frac{dz}{dx}$; (iii) y ; (iv) $\frac{dy}{dx}$; hence show that $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$.

TURNING POINTS.

A function $f(x)$ is called an “*increasing function*” of x for any range of values of x in which, as x increases, the value of $f(x)$ also increases.

It is called a “*decreasing function*” of x if, as x increases, the value of $f(x)$ decreases.

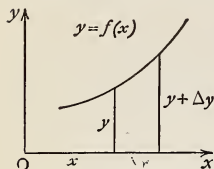


FIG. 23.

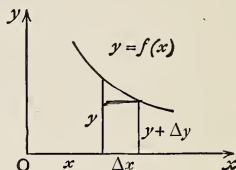


FIG. 24.

Fig. 23 represents an increasing function.

Fig. 24 represents a decreasing function.

If Δx is positive, then Δy is positive for an increasing function (Fig. 23), and Δy is negative for a decreasing function (Fig. 24).

Since $\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$, we see that $\frac{dy}{dx}$ is positive for an increasing function, and is negative for a decreasing function.

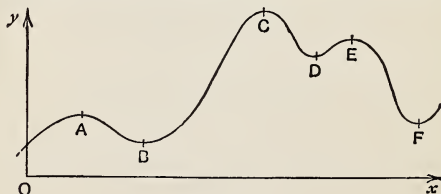


FIG. 25.

Fig. 25 represents a function which is an increasing function for some ranges of values of x , and a decreasing function for others.

Thus $f(x)$ is a decreasing function from A to B , and an increasing function from B to C .

The separating point B is called a *turning point*, and the value of $f(x)$ corresponding to the point B is called a *turning value* of the function.

There are two kinds of turning points.

$A, C, E, \dots$ corresponds to values of x for which the function is greater than at any other point near it: at these points the function is said to be a *maximum*.

$B, D, F, \dots$ correspond to values of x for which the function is less than at any other point near it: at these points the function is said to be a *minimum*.

At each of the turning points in Fig. 25, $\frac{dy}{dx} = 0$, for the function is neither increasing (i.e. $\frac{dy}{dx}$ positive) nor decreasing (i.e. $\frac{dy}{dx}$ negative).

EXERCISE IV. c.

1. Fig. 26 is the graph of $y = x^2$. Is y an increasing or decreasing function (i) from A to O , (ii) from O to B ? Is $\frac{dy}{dx}$ positive or negative (i) from A to O , (ii) from O to B ? Has y a maximum or minimum value anywhere?

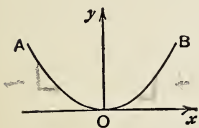


FIG. 26.

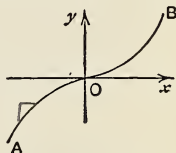


FIG. 27.

2. Fig. 27 is the graph of $y = x^3$. Answer the same questions as in Ex. 1.

3. Fig. 28 is the graph of $y = 4 + 3x - x^2$. Is y an increasing or decreasing function from (i) A to B , (ii) B to C , (iii) C to D , (iv) D to E , (v) E to F ? What are the signs of $\frac{dy}{dx}$ for these five portions? Has y a maximum or minimum value anywhere?

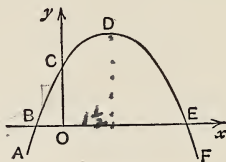


FIG. 28.

4. Draw freehand the graph of a function $y = f(x)$ which starts from the origin O , and is such that

- (i) from O to A , y is a decreasing function ;
- (ii) from A to B , $\frac{dy}{dx}$ is positive and y is negative ;
- (iii) from B to C , y is a positive increasing function ;
- (iv) from C to D , $\frac{dy}{dx}$ is negative and y is positive ;
- (v) from D to E , $\frac{dy}{dx}$ is negative and y is negative ;
- (vi) from E to F , $\frac{dy}{dx}$ is positive.

Has the function any maximum or minimum values ?

5. Draw freehand the graph of a function $y = f(x)$ for which

- (i) x is negative, y is negative, $\frac{dy}{dx}$ is positive ;
- (ii) x is negative, y is negative, $\frac{dy}{dx}$ is negative ;
- (iii) x is negative, y is positive, $\frac{dy}{dx}$ is positive ;
- (iv) x is negative, y is positive, $\frac{dy}{dx}$ is negative.

6. (i) Draw freehand the graph $ABCDE$ of a function

$y = f(x)$ such that the values of $\frac{dy}{dx}$ (or the gradients of the graph) at A, B, C, D, E are respectively $1, \frac{1}{2}, 0, -\frac{1}{2}, -1$.

(ii) What kind of a point is C ?

(iii) Is $\frac{dy}{dx}$ an increasing or decreasing function for the arc AC and the arc CE ?

(iv) What is the sign of $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d^2y}{dx^2}$ for points on the arc AC and the arc CE ?

7. (i) Draw freehand the graph $ABCDE$ of a function

$y = f(x)$ such that the values of $\frac{dy}{dx}$ (or the gradients of the graph) at A, B, C, D, E are respectively $-1, -\frac{1}{2}, 0, \frac{1}{2}, 1$.

(ii) What kind of a point is C ?

(iii) Is $\frac{dy}{dx}$ an increasing or decreasing function for the arc AC and the arc CE ?

(iv) What is the sign of $\frac{d^2y}{dx^2}$ for points on the arc AC and the arc CE ?

8. (i) Draw freehand the graph $ABCDE$ of a function

$y = f(x)$ such that the values of $\frac{dy}{dx}$ (or the gradients of the graph) at A, B, C, D, E are respectively $1, \frac{1}{2}, 0, \frac{1}{2}, 1$.

(ii) Is C a turning point ? *no. function increases.*

(iii) Is $\frac{dy}{dx}$ an increasing or decreasing function for the arc AC and the arc CE ?

(iv) What is the sign of $\frac{d^2y}{dx^2}$ for points on the arc AC and the arc CE ?

9. Answer the various questions in Ex. 8, taking the values of $\frac{dy}{dx}$ at A, B, C, D, E to be respectively $-1, -\frac{1}{2}, 0, -\frac{1}{2}, -1$.

10. AB , CD , EF , GH are portions of the graph of $y = f(x)$.

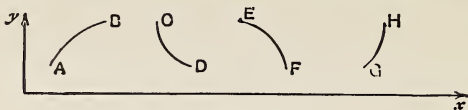


FIG. 29.

What can you say about the signs of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the arcs

- (i) AB ; (ii) CD ; (iii) EF ; (iv) GH ?

11. Can one of the minimum values of a function be greater than one of its maximum values? Illustrate by a figure.

12. Can a function have (i) exactly one minimum and two maximum values, (ii) exactly one minimum and three maximum values? Illustrate by a figure.

SUMMARY OF RESULTS.

For the function $y = f(x)$,

(i) if $\frac{dy}{dx} = 0$, and changes sign, there is a maximum or a minimum.

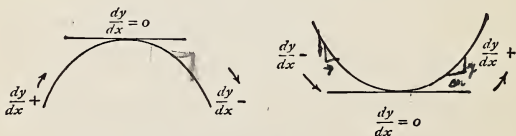


FIG. 30.

(ii) $\frac{dy}{dx}$ changes from $+$ to $-$ in passing through a maximum;

$\frac{dy}{dx}$ changes from $-$ to $+$ in passing through a minimum;

(iii) If $\frac{dy}{dx} = 0$ but does not change sign, there is neither a maximum nor a minimum.

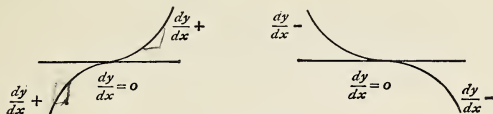


FIG. 31.

MAXIMA AND MINIMA PROBLEMS.

Example VI. The strength of a beam of uniform rectangular section varies as the breadth and the square of the depth. Find the breadth of the strongest rectangular beam that can be cut from a cylindrical tree-trunk of diameter 20 inches.

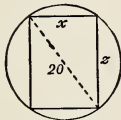


FIG. 32.

Let the breadth be x inches and the depth z inches ;

$$\begin{aligned} \therefore x^2 + z^2 &= 20^2 \quad (\text{Pythagoras}) \\ &= 400. \end{aligned}$$

Now the strength varies as xz^2

$$\begin{aligned} &= kxz^2 = kx(400 - x^2), \text{ where } k \text{ is a constant} \\ &= k(400x - x^3) ; \end{aligned}$$

$\therefore$ the function $y = 400x - x^3$ is to be a maximum.

$$\frac{dy}{dx} = 400 - 3x^2 ;$$

$$\therefore \frac{dy}{dx} = 0 \text{ if } 400 - 3x^2 = 0 \quad \text{or} \quad x^2 = \frac{400}{3} = 133.3 ;$$

$$\therefore x = \pm 11.54.$$

Since x inches is the breadth, x is positive and accordingly $x = -11.54$ is not in the range under discussion.

At $x = +11.54$, $\frac{dy}{dx} = 0$ and changes sign from $+$ to $-$.

$\therefore x = 11.54$ gives a maximum value for y ;

$\therefore$ the required breadth is 11.5 inches.

Check by drawing the graph of $y = 400x - x^3$.

EXERCISE IV. d.

1. Find the values of x which correspond to turning values of the following functions; determine whether they are maxima or minima; and sketch roughly the graphs of the functions:

(i) $x^2 - 2x$; (ii) $x^2 + 4x + 3$; (iii) $3 + 8x - 10x^2$;

(iv) $x + \frac{1}{x}$; (v) $x - \frac{1}{x}$; (vi) $x^4 - 4x$;

(vii) $x^4 - x^3$; (viii) $x^3 - x^2 - x + 1$;

(ix) $x^3 - 3x^2 - 9x + 7$; (x) $x^3 - 3x^2 + 3x - 1$.

2. Find the area of the largest rectangular piece of ground that can be enclosed by 200 hurdles each 4 feet long.

3. The parcel post regulations require that the sum of the length and girth of a parcel shall not exceed 6 feet. Find the volume of the largest box with a square base that can be sent by post.

4. A closed rectangular cistern is to be constructed to contain 80 cu. feet. It is to be 5 feet long. Find the breadth when the total area of its surface is a minimum.

5. The strength of a rectangular beam varies as the breadth and the square of the depth. Find the breadth of the strongest rectangular beam which has a perimeter of 4 feet.

6. Find the area of the largest rectangular piece of ground that can be enclosed by 200 hurdles each 4 feet long, if an existing fence is utilised to form one side.

7. A box without a lid is to be made from a sheet of metal of negligible thickness and is to have square ends; it is to hold $4\frac{1}{2}$ cu. feet. What is the least area of metal required?

8. An open gutter of rectangular section is formed out of a long rectangular strip of sheet iron 9 feet wide. Find the maximum area of the cross-section.

9. A rectangular sheet of cardboard is 8" long and 5" wide. Equal squares are cut out at each of the corners and the remainder is folded so as to form an open box. Find the maximum volume of the box.

10. Given $\pi r^2 h = 5$, find the value of r for which $2\pi r^2 + 2\pi r h$ is a minimum. What geometrical problem corresponds to this question?

11. A particle projected in a resisting medium is finally brought to rest : it travels s feet in t seconds, where $s = 6t - \frac{1}{2}t^3$. How far does it go?

12. Using the data of Ex. 3, find the volume of the largest circular cylinder that can be sent by parcel post.

13. A cylindrical vessel is open at one end and closed at the other ; for a given surface, prove that the volume is greatest if its height equals the radius of the base.

14. A skeleton box with two square ends is formed with 12 pieces of wire, and four other pieces of wire form an equal square round the middle of it. The total amount of wire available is one yard. What is the maximum volume of the box?

Take w
given
20 yds

APPROXIMATION.

Fig. 33 represents the graph of $y = f(x)$.

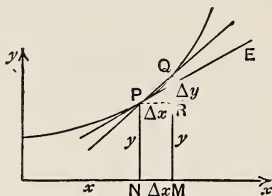


FIG. 33.

With the usual notation,

the gradient of the secant PQ is $\frac{\Delta y}{\Delta x}$,

the gradient of the tangent PE is $\frac{dy}{dx}$.

These are obviously not equal to each other.

But $\frac{dy}{dx}$ is the limit of $\frac{\Delta y}{\Delta x}$ when $\Delta x \rightarrow 0$.

And we may say that

$$\frac{dy}{dx} \simeq \frac{\Delta y}{\Delta x}, \text{ when } \Delta x \text{ is small,}$$

$$\Delta y \simeq \frac{dy}{dx} \times \Delta x, \text{ when } \Delta x \text{ is small.}$$

The smaller Δx is, the nearer the gradient of the chord PQ is to the gradient of the tangent PE .

Example VIII. If $y = x^2$, what is the error in the approximation $\Delta y \simeq \frac{dy}{dx} \times \Delta x$?

Now $y + \Delta y = (x + \Delta x)^2$;
 $\therefore y + \Delta y = x^2 + 2x \cdot \Delta x + (\Delta x)^2$,
 but $y = x^2$;
 $\therefore \Delta y = 2x \cdot \Delta x + (\Delta x)^2$.

But $\frac{dy}{dx} = \frac{d}{dx}(x^2) = 2x$;
 $\therefore \Delta y = \frac{dy}{dx} \times \Delta x + (\Delta x)^2$;
 $\therefore \Delta y \simeq \frac{dy}{dx} \times \Delta x$ with error $(\Delta x)^2$.

The relative size of the error may be seen in a figure.

If $y = x^2$, y sq. inches is the area of a square of side x inches, and $(y + \Delta y)$ sq. inches is the area of a square of side $(x + \Delta x)$ inches.

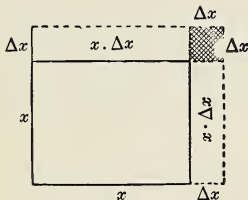


FIG. 34.

In taking Δy equal to $\frac{dy}{dx} \times \Delta x$ or $2x\Delta x$, we are supposing its value is the sum of the two rectangles and neglecting the small shaded square in Fig. 34.

Suppose x is 1" and Δx is 0.1", the error is 0.01 sq. in.

EXERCISE IV. e.

1. Find the error in the relation $\Delta y \simeq \frac{dy}{dx} \times \Delta x$ if

(i) $y = 5x^2$; (ii) $y = x^3$.

2. The area of an isosceles right-angled triangle of side x inches is A sq. inches.

(i) Find geometrically the value of ΔA in terms of Δx .

(ii) Express A in terms of x , and evaluate $\Delta A - \frac{dA}{dx} \times \Delta x$.

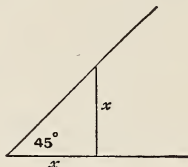


FIG. 35.

3. A body travels s feet in t sec., where $s = 10t - \frac{1}{5}t^3$; find an approximate expression (i) for Δs in terms of Δt , (ii) for the distance it moves in the interval of time $t = 2$ to $t = 2.1$.

4. A gas at constant temperature under a pressure of p lb. per sq. inch occupies v cu. inches, where $p v = c$ (a constant); if the pressure is increased from p to $p + \Delta p$, find an approximate expression for the change of volume.

5. Two telegraph posts of equal height are at a distance

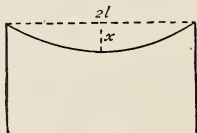


FIG. 36.

$2l$ feet apart. When the sag at the mid-point is x feet, the length of wire exceeds the distance between the poles by y feet, where $y = \frac{4x^2}{3l}$. Find an approximate value for Δy in terms

of Δx . What does this mean? If the distance between the poles is 30 yards, what is the effect of increasing the sag from 10 inches to 11 inches?

6. At sea-level, water boils at 212° F. At a height h feet above sea-level, the boiling-point is lowered t degrees, where $h = 520t + t^2$; find approximately the difference of heights of two places where the boiling-points are 200° and 201° F. respectively.

7. Fig. 37 represents the graph of $y = x^2$, unit 1" on each axis. If $ON = x$, it can be proved that the area bounded by ON , NP and the arc OP is $A = \frac{1}{3}x^3$ sq. inch. Find approximately ΔA in terms of Δx , and interpret the result geometrically.

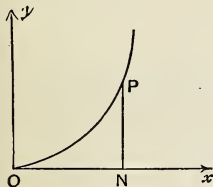


FIG. 37.

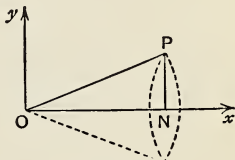


FIG. 38.

8. In Fig. 38 OP is the graph of $y = \frac{1}{2}x$; a circular cone is obtained by revolving OP about Ox . If $ON = x$ and the volume of the cone is V , prove that $V = \frac{1}{12}\pi x^3$. Find approximately ΔV in terms of Δx , and interpret the result geometrically.

9. If $y = \sqrt{z}$ and $z = \sqrt{1 - x^2}$, (i) prove that $\Delta y \simeq \frac{1}{2\sqrt{z}}\Delta z$, (ii) express Δz in terms of Δx , (iii) express Δy in terms of x and Δx , (iv) hence find $\frac{dy}{dx}$, i.e. $\frac{d}{dx} \sqrt{1 - x^2}$. Draw the graph of $y = \sqrt{1 - x^2}$, and so check your result.

Compare Ex. III. c, 17; also Ex. IV, b, 41 and 42. Note Ex 15 below.

10. As in Ex. 9 for $y = \pm \frac{1}{2}\sqrt{z}$ and $z = 4 - x^2$. Draw the graph of the ellipse $y = \pm \frac{1}{2}\sqrt{4 - x^2}$, and check.

11. Find $\frac{d}{dx}\{\pm \frac{1}{3}\sqrt{x^2-4}\}$. *check ans in book.* Draw the graph of the hyperbola $y = \pm \frac{1}{3}\sqrt{x^2-4}$, and check.

12. Using Ex. 9, find the equation of the tangent to the semi-circle $y = \sqrt{1-x^2}$ at $(3/5, 4/5)$.

13. Using Ex. 10, find the equation of the tangent to the ellipse $y = \pm \frac{1}{2}\sqrt{4-x^2}$ at $(6/5, 4/5)$.

14. If $u = y^2$, and if y is a function of x , (i) find a relation between Δu , Δy , y ; (ii) hence express $\frac{du}{dx}$ in terms of $\frac{dy}{dx}$; (iii) prove that $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$

15. Find, if possible, in previous exercises, examples of circle, ellipse, and hyperbola. In each case note whether the x - and the y -units are equal or unequal. Compare Ex. III. c, 17.

16. In which problems of Chapters III. and IV. is the derivative a speed? What other meanings may the derivative have? Is the derivative always a rate of change?

SUMMARY OF RESULTS.

APPROXIMATIONS AND ERRORS.

(i) An approximate value for

$$f(x + \Delta x) - f(x) \text{ is } \frac{d}{dx}f(x) \times \Delta x.$$

(ii) If in calculating the value of $f(x)$ there is a small error Δx in the value substituted for x , the error in the result is approximately $\frac{d}{dx}f(x) \times \Delta x$.

THREE VARIABLES.

If y is a function of x and x is a function of t , the rate at which y changes is given by

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt},$$

for $\Delta y \simeq \frac{dy}{dx} \times \Delta x$ and $\Delta x \simeq \frac{dx}{dt} \times \Delta t$.

Compare Ex. IV. b, 41, 42, and Ex. IV. e, 9-14.

THE QUADRATIC FUNCTION, $ax^2 + bx + c$.

(i) The gradient, $\frac{dy}{dx}$, of the curve $y = ax^2 + bx + c$ is $2ax + b$; and $\frac{d^2y}{dx^2} = 2a$.

(ii) If $a > 0$, the curve $y = ax^2 + bx + c$ has at $x = -b/2a$ a lowest point, if $a < 0$, a highest point.

(iii) The equation of this curve can be written

$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}.$$

This equation gives readily the results in (ii), and also the symmetry of the curve.

(iv) These curves are *parabolas*. Find examples of such functions and such curves in Ex. II. c, d, e; III. c; IV. c, etc.

THE CUBIC FUNCTION, ax^3 .

(i) The gradient of the curve $y = ax^3$ is $3ax^2$, and $\frac{d^2y}{dx^2} = 6ax$.

(ii) If $a > 0$, the curve is steadily rising; if $a < 0$, steadily falling. In each case the curve has no highest point, no lowest point.

(iii) These curves are *cubic parabolas*. The curves in Ex. IV. d, 1, (viii)-(x), are also cubic parabolas. Find examples in Ex. II. c, III. c ; IV. c, d, e ; Chap. IV. Example VI., etc.

THE FUNCTION, a/x .

(i) For $x = 0$, there is no point on the curve $y = a/x$. The curve has two branches.

(ii) If $a > 0$, these branches are in quadrants III. and I., and in each quadrant the curve is falling ; if $a < 0$, quadrants II. and IV. and in each the curve is rising.

(iii) These curves are *hyperbolas*. Compare the graph of Boyle's Law, $pv = c$, Ex. IV. e, 4. The curve in Ex. IV. e, 11 is also an hyperbola. Find examples of such functions and such curves in Chap. II., Example I., Ex. II., a, c, e ; and Ex. III. c ; Chap. IV, Example II., etc.

THE POWER FUNCTION, ax^n .

We consider only the functions having $a > 0$, and the first quadrant.

(i) If $n > 0$, the curve $y = ax^n$ is rising ; if $n < 0$, falling.

(ii) If $n > 0$, y is said to vary directly as x^n ; if $n < 0$, to vary inversely as x^n . Compare Chap. I. Note the relation between this paragraph and the three preceding.

Find examples of power function, parabolas and hyperbolas in Chap. II.

CHAPTER V.

INTEGRATION.

Example I. If $\frac{dy}{dx} = x^2 + 7$, express y in terms of x .

We know that $\frac{d}{dx}x^3 = 3x^2$;

$$\therefore \frac{d}{dx}(\frac{1}{3}x^3) = x^2.$$

Also $\frac{d}{dx}(7x) = 7$;

$$\therefore \frac{d}{dx}(\frac{1}{3}x^3 + 7x) = x^2 + 7;$$

$\therefore$ one value of y is given by $y = \frac{1}{3}x^3 + 7x$.

But $\frac{d}{dx}(\frac{1}{3}x^3 + 7x - 99)$ is also $x^2 + 7$,

or $\frac{d}{dx}(\frac{1}{3}x^3 + 7x + 2\frac{1}{4})$ is also $x^2 + 7$,

because $\frac{d}{dx}$ (any constant) is 0.

$\therefore y = \frac{1}{3}x^3 + 7x + c$, where c is any constant, satisfies the given equation, and is called its *general solution*.

Note.—An equation involving $\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}$, etc., is called a *differential equation*: and the process of expressing y as a

function of x is called *solving* or *integrating* the differential equation.

Whenever a differential equation is integrated, an arbitrary constant enters into the result.

EXERCISE V. a.

1. Prove that $y = x^4 - 5$ is a solution of the differential equation $\frac{dy}{dx} = 4x^3$. What is the general solution ?

2. Prove that $y = \frac{1}{5}x^5 + \frac{1}{2}x^2 + 3$ is a solution of the differential equation $\frac{dy}{dx} = x^4 + x$. What is the general solution ?

3. The graph of a curve $y = f(x)$ is such that its gradient is everywhere equal to $2x$. (i) What is the differential equation of the curve ? (ii) Prove that $y = x^2$, $y = x^2 + 2$, $y = x^2 - 1$ are each solutions, and interpret this result geometrically by drawing the graphs.

4. Prove that $y = \frac{1}{12}x^4 + 97x - 13$ is a solution of the differential equation $\frac{d^2y}{dx^2} = x^2$. What is the general solution ?

5. Solve $\frac{dy}{dx} = x$.

~~6.~~ Solve $\frac{dy}{dx} = 3x^2 - 2x$.

7. Solve $\frac{dy}{dx} = 4$.

8. Solve $\frac{d^2y}{dx^2} = 0$.

9. (i) What is $\frac{d}{dx}x^5$?

(ii) Solve $\frac{dy}{dx} = x^4$.

(iii) Solve $\frac{dy}{dx} = 3x^4$.

10. (i) What is $\frac{d}{dx}x^8$?

(ii) Solve $\frac{dy}{dx} = x^7$.

~~(iii)~~ Solve $\frac{dy}{dx} = 5x^7 - 2$.

11. (i) What is $\frac{d}{dx}\left(\frac{1}{x}\right)$? (ii) Solve $\frac{dy}{dx} = \frac{1}{x^2}$.

(iii) Solve $\frac{dy}{dx} = \frac{4}{x^2}$.

12. (i) What is $\frac{d}{dx}\left(\frac{1}{x^4}\right)$? (ii) Solve $\frac{dy}{dx} = \frac{1}{x^5}$.

(iii) Solve $\frac{dy}{dx} = 1 - \frac{3}{x^5}$.

13. (i) What is $\frac{d}{dx}(\sqrt{x})$? (ii) Solve $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$.

(iii) Solve $\frac{dy}{dx} = x + \frac{3}{\sqrt{x}}$.

14. Solve $\frac{dy}{dx} = 5x^2 - 3x + 9$. 15. Solve $\frac{dy}{dx} = x^5 - x^2 + 3$.

16. Solve $\frac{dy}{dx} = x^3 + \frac{1}{x^3}$. 17. Solve $\frac{dy}{dx} = 5\sqrt{x}$.

18. If $\frac{dy}{dx} = x - 2$, prove that $y = \frac{1}{2}x^2 - 2x + c$, and find c if $y = 5$ when $x = 6$.

19. If $\frac{ds}{dt} = 50 - 32t$, and if $s = 48$ when $t = 2$, find s in terms of t .

20. If $\frac{d^2s}{dt^2} = 4t$, and if, when $t = 0$, $\frac{ds}{dt} = 5$ and $s = 8$, express s in terms of t .

21. If a body travels s feet in t seconds, its velocity after t seconds is equal to $\frac{ds}{dt}$. Given that $\frac{ds}{dt} = 3 + 4t$, and that it passed a point A after 5 seconds from the start, find its distance from A after another 5 seconds.

22. A stone is thrown horizontally with a velocity of 12 feet per sec. from the top of a tower O , 100 feet high ; Ox is horizontal and Oy vertical (see Fig. 39). If P is its position t seconds after starting, and if $ON = x$ feet, $PN = y$ feet, then x, y obey the laws $\frac{d^2x}{dt^2} = 0$; $\frac{d^2y}{dt^2} = 32$, neglecting air resistance.

- (i) Express x and y each in terms of t .
- (ii) Express y in terms of x .
- (iii) How long does it take to reach the ground ?
- (iv) How far from the foot of the tower does it strike the ground ?
- (v) What is the gradient of the path of the stone where it strikes the ground ?
- (vi) A man is standing on the ground 50 feet from the foot of the tower ; where is the stone when it appears to him to be coming straight at him ?

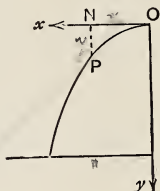


FIG. 39.

23. The rate at which a body cools is proportional to the excess of its temperature above that of the atmosphere. The temperature of a body is $T^\circ \text{C}$. at x minutes past one in a room of temperature $K^\circ \text{C}$. Express the law of cooling by a differential equation.

24. In climbing a mountain, the rate at which the atmospheric pressure p lb. per sq. in. decreases per unit increase of the height x feet is proportional to the pressure. Express this law by a differential equation.

Notation. The relation $\frac{dy}{dx} = f(x)$ is usually expressed in the form $y = \int f(x)dx$.

$$\therefore \Delta A < (y + \Delta y)\Delta x \quad \text{or} \quad \frac{\Delta A}{\Delta x} < y + \Delta y;$$

$$\therefore y < \frac{\Delta A}{\Delta x} < y + \Delta y,$$

$$\text{Lt. } y \leq \text{Lt. } \frac{\Delta A}{\Delta x} \leq \text{Lt. } (y + \Delta y),$$

$$y \leq \frac{dA}{dx} \leq y,$$

$$\frac{dA}{dx} = y = \frac{x^2}{5} + 1;$$

$$A = \frac{x^3}{15} + x + c.$$

$$\therefore A = \frac{x^3}{15} + x + c, \text{ where } c \text{ is the appropriate constant.}$$

But when $x = 0$, the area $A = 0$;

$$\therefore c = 0;$$

$$\therefore A = \frac{x^3}{15} + x.$$

Would there be any difference in the above argument if the graph sloped downwards instead of upwards?

Example III. Fig. 41 represents the graph of $y = \frac{x^2}{5} + 1$, not drawn to scale; if $OH = 2$, $OK = 7$, find the area bounded by AH , HK , KB and the arc AB .

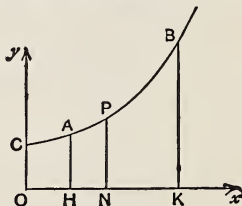


FIG. 41.

It is proved above in Example II. that if $ON = x$, the area of $CONP$ is A , where $\frac{dA}{dx} = \frac{x^2}{5} + 1$;

$$\therefore A = \int \left(\frac{x^2}{5} + 1 \right) dx$$

$$= \frac{x^3}{15} + x + c, \text{ where } c \text{ is the appropriate constant;}$$

$$\therefore \text{ the area of } COKB = \frac{7^3}{15} + 7 + c,$$

$$\text{and the area of } COHA = \frac{2^3}{15} + 2 + c;$$

$$\begin{aligned} \therefore \text{ the area of } AHKB &= \left[\frac{7^3}{15} + 7 \right] - \left[\frac{2^3}{15} + 2 \right] \\ &= \frac{343-8}{15} + 5 = \frac{335}{15} + 5 = \frac{67}{3} + 5 = 27\frac{1}{3}. \end{aligned}$$

Note the disappearance of the constant c , owing to the subtraction.

Notation. The symbol $\int_a^b f(x)dx$ is used to express the value of $\int f(x)dx$ when $x=b$ minus the value of $\int f(x)dx$ when $x=a$.

Thus

$$\int_2^7 x^2 dx \equiv \left[\frac{x^3}{3} \right]_{x=7} - \left[\frac{x^3}{3} \right]_{x=2} = \frac{7^3}{3} - \frac{2^3}{3} = \frac{343-8}{3} = \frac{335}{3},$$

$$\text{and we often write } \int_2^7 x^2 dx \equiv \left[\frac{x^3}{3} \right]_2^7.$$

Looking back at Example III. we see that the area of $AHKB$ can be written in any of the following forms :

$$\int_2^7 y dx \equiv \int_2^7 \left(\frac{x^2}{5} + 1 \right) dx \equiv \left[\frac{x^3}{15} + x \right]_2^7.$$

The expression $\int f(x)dx$ is called an *indefinite* integral because its value contains an arbitrary constant.

The expression $\int_a^b f(x)dx$ is called a *definite* integral because the arbitrary constant of integration disappears automatically.

Examples II., III. show that $\int_a^b f(x)dx$ is the area bounded by two ordinates $x=a$, $x=b$, the x -axis, and the graph of $y=f(x)$.

EXERCISE V. b.

1. Fig. 42 represents the graph of $y = x^2$; $ON = x$; (i) find in terms of x the area bounded by ON , NP , arc OP ; (ii) find the area $AHKB$, where $OH = 1$, $OK = 2$; (iii) express this area as a definite integral.

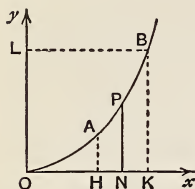


FIG. 42.

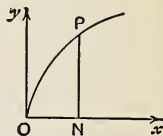


FIG. 43.

2. Fig. 43 represents the graph of $y^2 = x$; $ON = x$; find the area ONP .

3. Interpret geometrically $\int_1^3 x \, dx$ and evaluate it (i) by direct calculation, (ii) geometrically.

4. Fig. 44 represents part of the graph of $y = (x - 1)(4 - x)$; (i) find OA and OB ; (ii) find the area between Ox and the portion of the curve above Ox .

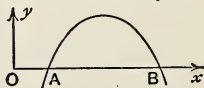


FIG. 44.

5. Draw the graph of $\frac{1}{x^2}$ and find the area (i) between it and the ordinates $x = 1$, $x = 2$ and the x -axis; (ii) between it and the lines $y = 1$, $y = 2$ and the y -axis. Check each result by "counting squares."

6. If Fig. 42 represents the graph of $y = x^3$; prove that the area BLO is three times the area BKO .

7. Find the values of

(i) $\int_1^2 (x^2 + 5)dx$;

(ii) $\int_2^4 \frac{dx}{x^3}$;

(iii) $\int_{-1}^{+1} (x + x^2)dx$;

(iv) $\int_0^{\frac{1}{2}} (1+x)^2 dx$.

8. Fig. 45 represents a solid formed by rotating part of the graph of $y = x^2$ about Ox .

$ON = x$, $NP = y$, $OM = x + \Delta x$, $QM = y + \Delta y$, $OH = \frac{1}{4}$, $OK = \frac{3}{4}$.

The figure shows cross-sections perpendicular to Ox .

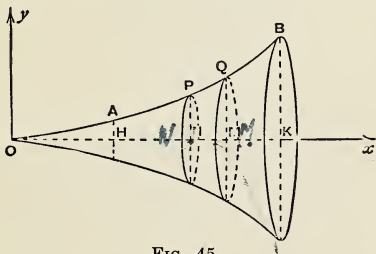


FIG. 45.

V is the volume of the solid between O and the cross-section through P .

(i) What is the area of the cross-section through P ?

(ii) What is the area of the cross-section through Q ?

(iii) Interpret $\pi y^2 \Delta x$ geometrically.

(iv) Prove that $\pi(y + \Delta y)^2 \cdot \Delta x > \Delta V > \pi y^2 \cdot \Delta x$.

(v) Prove that $V = \pi \int_0^x y^2 dx$.

(vi) Express V in terms of x .

(vii) Find the volume between the cross-sections at A and B , and express it also as a definite integral.

(viii) If the graph of $y = f(x)$ is drawn, interpret geometrically $\int_0^x \pi y^2 dx$.

(ix) If the graph of $y = f(x)$ is drawn, interpret geometrically $\int_0^y \pi x^2 dy$.

9. (i) Draw the graph of $y = \frac{1}{3}x$; take a point P on it and draw PN perpendicular to Ox .
 (ii) What solid is formed by revolving OP about Ox ?
 (iii) Find, by the method of Ex. 8, the volume of the solid between O and the section through P perpendicular to Ox , in terms of x .

✓10. Fig. 46 represents a section of a hemisphere of radius of 10 inches, centre O ; $ON = x$ inch.

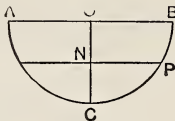


FIG. 46.

- (i) Find PN in terms of x .
 ✓(ii) Find the area of the cross-section of the hemisphere at N perpendicular to ON .
 (iii) Find an approximate expression for the volume of the part of the hemisphere contained between two parallel planes at distances x and $x + \Delta x$ from O .
 ✓(iv) If the volume between the sections through O and through N , perpendicular to ON , is V cu. inch, express V in terms of x .
 (v) Express as a definite integral the volume of the spherical segment between C and the section through N ; and simplify it.
 (vi) Find the volume of the hemisphere.
- ✓11. What solid is generated by the revolution of a circle about its diameter. Find its volume by integration.
12. (i) Sketch the graphs of $y^2 = x$ and $x^2 = 8y$ for positive values of x .
 (ii) Show that they cross each other at the point $(4, 2)$.
 (iii) Show that the area of the portion common to both is $\int_0^4 \left(\sqrt{x} - \frac{x^2}{8} \right) dx$, and evaluate it. Check by "counting squares."

13. Find the volume of the solid generated by the revolution

Geom. of the ellipse $\frac{x^2}{36} + \frac{y^2}{16} = 1$, (i) about the x -axis,
(ii) about the y -axis.

14. (i) Sketch the half of the ellipse $\frac{x^2}{4} + y^2 = 81$ which lies
between $x = 0$ and $x = 18$.

Geom. (ii) This portion is rotated about Ox to form the explosive
head of a torpedo; unit on each axis, 1 inch.
Find the volume of the explosive in the head.

CHAPTER VI.

SEQUENCES AND SERIES.

Example I. The following sequence of numbers obeys a simple law ; find (i) the 15th term, (ii) the n th term in the sequence

$$11, 15, 19, 23, 27, 31, \dots$$

We see that each term exceeds the term before it by 4.

$$\text{The 2nd term} = 15 = 11 + 4.$$

$$\text{The 3rd term} = 19 = 11 + 4 + 4 = 11 + 2 \times 4.$$

$$\text{The 4th term} = 23 = 11 + 4 + 4 + 4 = 11 + 3 \times 4, \\ \text{and so on ;}$$

$$\therefore \text{ the 15th term} = 11 + 14 \times 4 = 11 + 56 = 67, \\ \text{and the } n\text{th term} = 11 + (n-1) \times 4 = 11 + 4n - 4 = 4n + 7.$$

Example II. Find the n th term in each of the following sequences :

$$(i) \quad 7, 21, 63, 189, \dots ;$$

$$(ii) \quad \frac{3}{4}, \frac{9}{7}, \frac{27}{16}, \frac{81}{13}, \dots$$

(i) We see that each term is three times the term before it.

$$\text{The 2nd term} = 21 = 7 \times 3.$$

$$\text{The 3rd term} = 63 = 7 \times 3 \times 3 = 7 \times 3^2.$$

$$\text{The 4th term} = 189 = 7 \times 3 \times 3 \times 3 = 7 \times 3^3, \\ \text{and so on ;}$$

$$\therefore \text{ the } n\text{th term} = 7 \times 3^{n-1}.$$

(ii) We notice that the numerators are $3, 3^2, 3^3, 3^4, \dots$;
 $\therefore$ the numerator of the n th term is 3^n .

The denominators are $4, 4 + 3, 4 + 2 \times 3, 4 + 3 \times 3, \dots$;
 $\therefore$ the denominator of the n th term is

$$4 + (n - 1) \times 3 = 4 + 3n - 3 = 3n + 1;$$

$$\therefore \text{the } n\text{th term} = \frac{3^n}{3n + 1}.$$

EXERCISE VI. a.

1. The following sequences of numbers obey simple laws ; write down in each case the 6th and the n th number in the sequence :

- (i) $2, 4, 6, 8, \dots$; (ii) $1, 4, 9, 16, 25, \dots$;
 (iii) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots$; (iv) $10, 200, 3000, 40000, \dots$;
~~(v) $3, 6, 12, 24, \dots$;~~ (vi) $\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{7}{16}, \frac{9}{32}, \dots$;
 (vii) $5, 12, 19, 26, 33, \dots$; (viii) $10, 7, 4, 1, -2, \dots$;
 (ix) $0.1, 0.02, 0.003, 0.0004, \dots$;
 (x) $-1, 1, -1, 1, -1, \dots$;
 (xi) $8, 27, 64, 125, \dots$; (xii) $14, 108, 1012, 10016, \dots$

2. Write down the first three terms of the sequence in which the n th term is

- (i) $2n - 1$; (ii) $\frac{3^n}{18}$; (iii) $\frac{n + 1}{n + 2}$;
 (iv) $n^2 + n$; (v) $3 \cdot 10^{n-1} + 5n$; (vi) $n^3 - 1$.

3. The sum of the first n terms of a sequence is $\frac{n(n + 1)}{2}$;

(i) what is the 1st term ? ; (ii) what is the sum of the first two terms ? ; (iii) what is the 2nd term ? ; (iv) what is the 10th term ? ; (v) what is the n th term ?

4. The sum of the first n terms of a sequence is $\frac{n(n+1)(2n+1)}{6}$; find (i) the 1st term, (ii) the 4th term,

(iii) the n th term. *Same as I (4)*

5. What is the average of the following 14 numbers ?

$3, 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, 47, 51, 55.$

Is it necessary to add them all up ?

When you know their average, how can you find their sum ?

What is it ?

6. Find (i) the average, (ii) the sum of the following sequences :

- (i) 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, 38, 41, 44, 47, 50.
- (ii) 60, 55, 50, 45, 40, 35, 30, 25, 20, 15, 10.
- (iii) 20, 13, 6, -1, -8, -15, -22, -29, -36, -43, -50.
- (iv) $3\frac{1}{2}$, $4\frac{3}{4}$, 6, $7\frac{1}{4}$, $8\frac{1}{2}$, $9\frac{3}{4}$, 11, $12\frac{1}{4}$, $13\frac{1}{2}$, $14\frac{3}{4}$, 16.

7. How many terms are there in the following sequences ?

- (i) 7, 9, 11, 13, 15, ... , 53, 55, 57, 59 ; *middle term 53.*
- (ii) 22, 29, 36, 43, 50, 57, ... , 155, 162, 169 ;
- (iii) 100, 97, 94, 91, 88, ... , 46, 43, 40 ;
- (iv) $3\frac{1}{6}$, $4\frac{1}{8}$, $5\frac{1}{10}$, $6\frac{1}{12}$, ... , $20\frac{1}{40}$, $21\frac{1}{42}$.

8. Find the least number of three digits which belongs to the sequence :

- (i) 6, 13, 20, 27, ... ; *Find nth term*
- (ii) 1, 2, 4, 8, 16, ... ; *subst values*
- (iii) 3, 14, 25, 36, ... *e.g. 1) $7n-1$
Subst $n=14, 15$*

Find also the n th term in each case.

9. In a sequence the terms increase by a constant amount (positive or negative) [as, for example, in 6 (i), (iii)]; find the 2nd term if

- (i) the 1st term is 8 and the 10th term is 71 ; *$8 + (n-1)d = 71$*
- (ii) the 1st term is 3 and the 15th term is 4 ;
- (iii) the 1st term is 14 and the 8th term is -7 ;
- (iv) the 4th term is 25 and the 12th term is 97.

10. Find the sum of the following :

- (i) $25 + 28 + 31 + 34 + \dots$ *Two series last 25 to 100 and 97 to 49.*
 $+ 94 + 97 + 100 + 97 + 94 + 91 + \dots + 52 + 49 ;$
- (ii) $4 - 5 + 6 - 7 + 8 - 9 + \dots + 40 - 41 + 42$
Sum of even then sum of odd.

11. Find the 10th term in the series

$$1 + \frac{13}{1}b + \frac{13 \cdot 12}{1 \cdot 2}b^2 + \frac{13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3}b^3 + \dots,$$

and compare the binomial expansions in Chap. VIII.

$$25 + 3(n-1) = 100 \therefore n = 26, \text{ Avg} = 62.5$$

$$77 + (-2)(n-1) = 49 \therefore n = 17 \text{ Avg} = 73.$$

ARITHMETICAL PROGRESSIONS.

Definition. A sequence of numbers in which each term exceeds the preceding term by the same amount, positive or negative, is called an *Arithmetical Progression* or, for short, an A.P.

E.g. (i) 5, 12, 19, 26, 33, ...,
 or (ii) 10, 2, - 6, - 14, ... ,
 or (iii) $6\frac{1}{4}$, $7\frac{3}{4}$, $9\frac{1}{4}$, $10\frac{3}{4}$, ...

In (i) each term exceeds the term before it by 7.

This amount 7 is called the *common difference*.

In (ii) the common difference is $2 - 10 = - 8$,
 or $- 6 - 2 = - 8$.

In (iii) the common difference is $7\frac{3}{4} - 6\frac{1}{4} = 1\frac{1}{2}$.

If three numbers are in arithmetical progression, the middle number is called the *arithmetic mean* of the other two.

E.g. 21, 29, 37 are in A.P. ;

$\therefore$ 29 is the arithmetic mean of 21 and 37.

The arithmetic mean of two numbers is simply their average.

Example III. Find the sum of 24 terms of the sequence

$$- 41, - 33, - 25, - 17, \dots$$

The common difference $= - 33 - (- 41) = - 33 + 41 = 8$;

$\therefore$ the 24th term $= - 41 + 23 \times 8 = - 41 + 184 = 143$;

$\therefore$ the sum $s = - 41 - 33 - 25 \dots + 127 + 135 + 143$.

The average of the terms of this sequence is

$$\frac{- 41 + 143}{2} = \frac{102}{2} = 51 ;$$

$$\begin{aligned} \therefore s &= 24 \times 51 \\ &= 1224. \end{aligned}$$

Another method.

$$s = -41 - 33 - 25 - \dots + 127 + 135 + 143.$$

Write the sequence backwards ;

$$\therefore s = 143 + 135 + 127 + \dots - 25 - 33 - 41 ;$$

$$\therefore \text{adding} \quad 2s = 102 + 102 + 102 + \dots + 102.$$

But there are 24 terms ;

$$\therefore 2s = 102 \times 24 ;$$

$$\therefore s = \frac{102 \times 24}{2} = 51 \times 24 = 1224.$$

Example IV. The 10th term of an A.P. is 17 and the 25th term is 41 ; find the 2nd term.

Let the 1st term be a and the common difference d ;

$$\therefore \text{the 10th term is } a + 9d ;$$

$$\therefore a + 9d = 17.$$

Similarly,

$$a + 24d = 41 ;$$

$$\therefore -15d = -24 ;$$

$$\therefore d = \frac{24}{15} = \frac{8}{5} = 1\frac{3}{5} ;$$

$$\therefore a = 17 - 9d = 17 - \frac{72}{5} = 17 - 14\frac{2}{5} = 2\frac{3}{5} ;$$

$$\therefore \text{the 2nd term} = a + d = 2\frac{3}{5} + 1\frac{3}{5} = 4\frac{1}{5}.$$

EXERCISE VI. b.

1. The 1st term of an A.P. is 10 and the 3rd is 11 ; find the 20th.

2. The 1st term of an A.P. is 11 and the 10th term is -4 ; find the 3rd term.

3. The 1st term of an A.P. is 21, the last term is 193, and the common difference is 2 ; find the number of terms.

4. Find the arithmetic mean of 23 and 90.

5. If 12, x , y , z , 41 are in A.P., find x , y , z . This question might be worded as follows : "insert three arithmetic means between 12 and 41."

Sum the series in Ex. 6-15 :

6. $7 + 8 + 9 + \dots + 100.$

7. $22 + 19 + 16 + \dots$ 20 terms.

8. $5.3 + 7.2 + 9.1 + \dots$ 25 terms.

9. $8\frac{1}{3} + 6\frac{5}{8} + 5\frac{1}{3} + \dots$ 18 terms.

10. $3 + 7 + 11 + 15 + \dots$ n terms.

11. $a + (a + d) + (a + 2d) + (a + 3d) + \dots$ 12 terms.

12. $a + (a + d) + (a + 2d) + \dots$ n terms.

13. $11 + 13\frac{1}{2} + \dots + 36$.

14. $23 + \dots + 67$ (12 terms).

15. $a + \dots + l$ (n terms).

16. In the sequence

$[1]$; $[2, 3]$; $[4, 5, 6]$; $[7, 8, 9, 10]$; $[11, 12, 13, 14, 15]$, ..., what is the first number in (i) the 8th bracket, (ii) the n th bracket?

17. Find the sum of all numbers less than 100 which are not divisible by 7.

18. The first two terms of an A.P. are a, x ; what is (i) the 3rd term, (ii) the n th term? *See Ans. book.*

19. In the sequence

$[1]$; $[3, 5]$; $[7, 9, 11]$; $[13, 15, 17, 19]$; ..., *[nth bracket]* what is the first number in (i) the 10th bracket, (ii) the n th bracket?

20. The n th term of a sequence is $5n - 11$; what is the common difference and the sum of the first n terms? *Subst $n=1$ obtain C.D. by rule*

21. (i) Each term of an A.P. is multiplied by 7; is the result an A.P.? *yes*

(ii) Each term of an A.P. is multiplied by itself; is the result an A.P.? *no*

22. What is the sum of all numbers divisible by 12 between 1000 and 5000? *1008 to 5008
last 4992*

23. In a sequence of n terms, the last term is l and the common difference is d ; what is the first term? *Let it be a .*

24. If a, b are integers, find the sum of all integers from a to b inclusive, assuming $a < b$.

*no. of no. between a & b : $b - a + 1$.
Average value : $\left[\frac{a + b + 1}{2} \right] = \frac{a + b}{2}$
 $\therefore S = \frac{a + b}{2} (b - a + 1)$*

SUMMARY OF RESULTS.

Any arithmetical progression can be written in the form

$$a, a + d, a + 2d, a + 3d, \dots;$$

(i) the n th term $= a + (n - 1)d$;

or, if l = the last term and n = the number of terms,

$$l = a + (n - 1)d;$$

(ii) the sum of n terms $= s = n \times$ average of the terms ;

or
$$s = \frac{n}{2}(a + l)$$

and
$$s = \frac{n}{2}[2a + (n - 1)d];$$

(iii) the arithmetic mean of a, b is $\frac{a + b}{2}$.

GEOMETRICAL PROGRESSIONS.

Definition. A sequence of numbers in which each term bears a fixed ratio to the term preceding it is called a *Geometrical Progression* or, for short, a G.P.

E.g. (i) 4, 12, 36, 108, 324, ...,
or (ii) 7, - 14, 28, - 56, 112, ... ,
or (iii) 12, 4, $1\frac{1}{3}$, $\frac{4}{9}$, $\frac{4}{27}$, ...

In (i) the ratio is $\frac{12}{4} = 3 = \frac{36}{12} = \frac{108}{36} = \dots$.

This amount 3 is called the *common ratio*.

In (ii) the common ratio $= \frac{-14}{7} = -2$.

In (iii) the common ratio $= \frac{4}{12} = \frac{1}{3}$.

If three numbers are in geometrical progression, the middle number is called the *geometric mean* of the other two.

E.g. 4, 12, 36 are in G.P. ;

$\therefore$ 12 is the geometric mean of 4 and 36.

If a, x, b are in G.P., $\frac{x}{a} = \frac{b}{x}$, by definition ;

$$\therefore x^2 = ab ; \text{ and } x = \pm \sqrt{ab}.$$

Therefore the two geometric means of a, b are $\pm \sqrt{ab}$.
Compare "mean proportional," definition and Ex. I. b.

Example V. Find the sum of 10 terms of the sequence
18, 12, 8, $5\frac{1}{3}$, ...

$$\text{The common ratio} = \frac{12}{18} = \frac{2}{3} ; \left[\frac{8}{12} = \frac{2}{3} \text{ and } \frac{5\frac{1}{3}}{8} = \frac{2}{3} \right] ;$$

$$\therefore \text{the 10th term} = 18 \times \left(\frac{2}{3}\right)^9.$$

$$s = 18 + 12 + 8 + \dots + 18 \times \left(\frac{2}{3}\right)^8 + 18 \times \left(\frac{2}{3}\right)^9 ;$$

$$\therefore \frac{2}{3}s = 12 + 8 + 5\frac{1}{3} + \dots + 18 \times \left(\frac{2}{3}\right)^9 + 18 \times \left(\frac{2}{3}\right)^{10}.$$

The terms from 12 up to $18 \times \left(\frac{2}{3}\right)^9$ are the same in both lines.

$$\therefore s - \frac{2}{3}s = 18 - 18 \times \left(\frac{2}{3}\right)^{10} = 18[1 - \left(\frac{2}{3}\right)^{10}] ;$$

$$\therefore 3s - 2s = 54[1 - \left(\frac{2}{3}\right)^{10}] ;$$

$$\therefore s = 54[1 - \left(\frac{2}{3}\right)^{10}].$$

An approximate value can be found by logarithms.

$$\frac{2}{3} = 0.6667 = 10^{1.8240} ;$$

$$\therefore \left(\frac{2}{3}\right)^{10} = 10^{2.240} = 0.0174 ;$$

$$\therefore s = 54 \times 0.9826 \\ = 53.1.$$

EXERCISE VI. c.

1. Write down, using the index notation, the n th terms in geometrical progressions which start :

(i) 7, 35 ;

(ii) 1, - 2 ;

(iii) 8, 4 ;

(iv) - 10, 6 ;

(v) 6, 7 ;

(vi) $2\frac{1}{3}$, 3 ;

(vii) - x, x^3 ;

(viii) a, b ;

(ix) $a, 1$.

2. Simplify (i) $8(-\frac{1}{2})^7$; (ii) $(-1)^{2n} - (-1)^{2n-1}$.

3. What is the 1st term in the sequence $1, \frac{3}{4}, \frac{9}{16}, \frac{27}{64}, \dots$, which is less than 0.001 ?

4. If $s = 3 + 6 + 12 + 24 + 48 + 96 + 192 + 384 + 768$, write down $2s$ in full ; then subtract, and so find s .

5. If $s = 54 + 18 + 6 + 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \frac{2}{243}$, write down $\frac{1}{3}s$ in full; then subtract, and so find s .

6. If $s = a + ar + ar^2 + ar^3 + ar^4 + ar^5 + ar^6$, write down rs in full; what is $rs - s$? Hence find s .

7. Sum the following sequences; do not multiply out the result:

- (i) $4, 4 \times 3, 4 \times 3^2, 4 \times 3^3, \dots$ to 10 terms; $4(3^4 - 1)$
- (ii) $5, 5 \times \frac{2}{5}, 5 \times (\frac{2}{5})^2, 5 \times (\frac{2}{5})^3, \dots$ to 12 terms; $5(1 - (\frac{2}{5})^{12})$
- (iii) $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}, \dots$ to 8 terms; $\frac{3}{4}(1 - (\frac{2}{3})^8)$
- (iv) $7, -14, 28, -56, \dots$ to 11 terms; $7(1 - (-2)^{11})$
- (v) $27, -12, 5\frac{1}{3}, \dots$ to 20 terms. $27(1 - (\frac{2}{3})^{20})$

8. (i) Divide $1 - r^6$ by $1 - r$.

(ii) What is the quotient when $1 - r^{10}$ is divided by $1 - r$?

(iii) Express $1 + r + r^2 + r^3 + \dots + r^{18} + r^{19} + r^{20}$ as a simple fraction.

(iv) Express $a + ar + ar^2 + ar^3 + \dots + ar^{20}$ as a simple fraction.

9. If $s = a + ar + ar^2 + \dots$ to n terms,

(i) what is the n th term? ar^{n-1}

(ii) what is the last-but-one term? ar^{n-2}

(iii) write down the value of rs , putting in the first three and the last three terms;

(iv) what is the value of $rs - s$? Hence find s .

10. (i) Prove that $1 + \frac{1}{10} + \frac{1}{100} + \dots$ to 8 terms $= \frac{10}{9} \left(1 - \frac{1}{10^8}\right)$. *Find value $s - s_8$*

(ii) What is the value of this answer to 3 places of decimals?

(iii) What is the approximate error per cent. in taking the sum of these 8 terms as $\frac{10}{9}$?

(iv) What is the sum of 50 terms of the same series, and what is the approximate error per cent. in taking the sum as $\frac{10}{9}$?

(v) Find an expression for the sum of 100 terms of the same series; for the sum of 10000 terms; of 1,000,000 terms.

(vi) Reduce to a common fraction each of the following: 1.1111111, 1.111... 1 to 50 figures, 1.111... 1 to 100 figures, etc. Compare parts (i), (iv) and (v) above. *Express in com. fractions.*

(vii) Prove that each of the series in (vi) has the sum $\frac{10}{9}\left(1 - \frac{1}{(10)^n}\right)$, where n is the number of terms (or figures).

(viii) What is the limit of $\frac{1}{10^n}$ as n increases without limit ?
of $\left(1 - \frac{1}{10^n}\right)$?

(ix) What is the limit of the sum of n terms of this series as n increases without limit ?

(x) By division, try to express $\frac{10}{9}$ as a decimal.

(xi) By definition, the endless series $1.111 \dots$ has the value $\frac{10}{9}$. The student should note that $\frac{10}{9}$ is not the sum of n terms of this series, however large n may be, but is the limit of the sum of n terms as n increases without limit. It is customary to call an endless series an "*infinite series*," and when there is a limit of the sum of n terms, to call that limit "*the sum to infinity of the series*."

11. AB is a straight line, one inch long. Bisect AB at C_1 ; bisect C_1B at C_2 ; bisect C_2B at C_3 ; bisect C_3B at C_4 ; and so on.

(i) What are the lengths of AC_1 , C_1C_2 , C_2C_3 , C_3C_4 , C_4C_5 , ... ?

(ii) What are the lengths of AC_2 , AC_3 , AC_4 , AC_5 , AC_{10} , AC_{100} ?

(iii) What is the limit of the length of AC_n when $n \rightarrow \infty$?

(iv) What is the difference between the sum of the first 10 terms and the sum of the first 50 terms of the sequence $8, 4, 2, 1, \frac{1}{2}, \dots$?

(v) What is a fair approximation for the sum of the first 10 terms of this sequence ?

(vi) What meaning would you give to the expression "the sum to infinity" of this sequence ?

12. Which of the following geometrical progressions has a "sum to infinity" ?

(i) $1, \frac{10}{9}, (\frac{10}{9})^2, (\frac{10}{9})^3, \dots$; (ii) $100, 90, 81, \dots$;
(iii) $4, -6, +9, \dots$; (iv) $1, -\frac{1}{2}, \frac{1}{4}, \dots$.

13. Sketch freehand a graph (it will consist of a set of disconnected points) representing the sum of the first 1, 2, 3, 4, ... terms of the sequence as a function of the number of terms

$$(i) \ 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots; \quad (ii) \ 1, 2, 4, 8, \dots$$

14. Express $0.4\dot{3}$ as a geometrical progression, and then find its "sum to infinity."

15. Find the "sum to infinity" of the following:

- (i) $500 + 400 + 320 + 256 + \dots$;
- (ii) $10, -2, 0.4, -0.08, \dots$;
- (iii) $a + ar + ar^2 + ar^3 + \dots$ if $-1 < r < 1$.

16. Find, using logarithms, an approximate value of

- (i) $50 + 50(1.05) + 50(1.05)^2 + \dots$ to 10 terms;
- (ii) $8 + 12 + 18 + \dots$ to 18 terms.

SUMMARY OF RESULTS.

Any geometrical progression can be written in the form

$$a, \ ar, \ ar^2, \ ar^3, \dots;$$

(i) the n th term $= ar^{n-1}$;

(ii) the sum of n terms $= \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}$;

(iii) if $-1 < r < +1$, there is a "sum to infinity,"

and it is $\frac{a}{1 - r}$;

(iv) the two geometric means of a, b are $\pm \sqrt{ab}$.

MISCELLANEOUS EXAMPLES ON SERIES.

EXERCISE VI. d.

1. A train travelling 12 m.p.h. gathers speed uniformly till five minutes later its speed is 40 m.p.h.; what was its speed at the end of the first minute, and how far did it go in the five minutes?

2. Thirty concentric circles are engraved at equal distances apart on a metal plate ; the radii of the inner and the outer circles are 4.5" and 7.3" ; find the distance between two consecutive circles correct to $\frac{1}{1000}$ inch.

3. The height of a tree is 20 feet and one year later is 24 feet ; its growth each year is $\frac{3}{4}$ of its growth the previous year. What is the limiting height of the tree, however long it lives ?

4. A square is divided into 25 equal squares, and the numbers 1, 2, 3, ... up to 25 are arranged in these squares so that the sum of the numbers in each row and each column is the same ; what is the sum of the numbers in each row ? [This arrangement is called a magic square.]

5. A pendulum is set swinging ; its first vibration is through 20° , and each succeeding vibration is $\frac{9}{10}$ of the one before it. What is the total angle through which it swings before coming to rest ?

6. What is the sum of the first n odd integers ?

7. If $s = 1 + 2x + 3x^2 + \dots + nx^{n-1}$, what is $s - xs$ equal to ? Hence find s .

8. The series

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots$$

is obviously greater than

$$1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots$$

or

$$1 + \frac{1}{2} + \left(\frac{2}{4}\right) + \left(\frac{4}{8}\right) + \dots$$

How many terms of the first series will give a sum greater than 100 ?

9. Show that the sum of any number of terms of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$ always lies between 1 and $\frac{1}{2}$.

10. Find the sum of 20 terms of the series

$$\frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \frac{1}{5 \times 6} + \dots$$

[Note that $\frac{1}{2 \times 3} = \frac{1}{2} - \frac{1}{3}$, etc.]

11. The sum of n terms of a series is n^3 ; find the first four terms.

GROWTH FUNCTIONS.

Example VI. If \$1 is invested at 3 per cent. compound interest, find its amount at the end of x years.

The interest on \$100 is \$3 ;

$\therefore$ the interest on \$1 is \$0.03 ;

$\therefore$ \$1 amounts to \$1.03 at the end of 1 year.

We therefore start the second year with \$1.03.

This amounts to $\$(1.03)(1.03)$ at the end of the second year ;

$\therefore$ the amount at the end of the second year is $\$(1.03)^2$.

Similarly, the amount at the end of the third year is $\$(1.03)^3$, and so on ;

$\therefore$ the amount at the end of x years is $\$(1.03)^x$.

Such a function as $(1.03)^x$ is called a *Growth function*, because it represents the magnitude of a quantity which increases in such a way that its growth over a definite period (in the above case one year) is proportional to its magnitude at the beginning of that period.

EXERCISE VI. e.

1. (i) Draw the graph of the growth function $(1.5)^x$ as x varies from 0 to 4.
- (ii) A plant grows so that its height is $(1.5)^x$ inches after $10x$ days, as long as $x < 4$; what is its height (1) to start with, (2) at the end of 4 days, 8 days, 12 days, 16 days ?
- (iii) In what ratio does its height increase (1) from the end of the 4th to the end of the 8th day, (2) from the end of the 8th day to the end of the 12th day, (3) from the end of the 12th day to the end of the 16th day ?

2. Assuming that the population of a country is a growth function of the time, find the population in 1900, given that the population in 1880 was 23.6 million and in 1890 was 27.4 million.

3. The excess of the temperature of a liquid cooling in a room of fixed temperature above that of the room is a growth function of the time. If the temperature of the room is 30° C. and if at 1 p.m. the temperature of the liquid is 70° C. and at 1.10 p.m. it is 60° C., what is its temperature at 1.30 p.m. ?

4. A merchant finds that his capital is a growth function of the time. After 10 years' work his capital is \$50,000, and after 20 years' work it is \$80,000 ; what will it probably be after 25 years' work ?

5. If $r^x = 10$ and $r^{x+4} = 20$, what is r^{x+8} and what is r ?

6. In an influenza epidemic it was found that the number of cases was a growth function of the time. In ten days the number had increased from 320 to 480 ; what was the daily increase per cent. ?

7. (i) If $f(x)$ is a growth function of x , prove that

$$\frac{f(x+a)}{f(x)} = \frac{f(x+2a)}{f(x+a)} = \frac{f(x+3a)}{f(x+2a)}, \text{ etc.}$$

(ii) Given a graph of a function, how would you test whether the function is or is not a growth function ?

(iii) Apply (i) and (ii) to the graphs of 2^x and 10^x (Revision Exercise B).

(iv) For comparison, prove that if $f(x)$ is a linear function of x , then

$$f(x+a) - f(a) = f(x+2a) - f(x+a) = \text{etc.}$$

(v) Find simple functions (compare Ex. II. e) which have the property that

$$f(3) - f(1) = f(5) - f(3) = f(7) - f(5).$$

(vi) Show that in (i) $f(x)$, $f(x+a)$, $f(x+2a)$... is a G.P. and that in (iii) $f(x)$, $f(x+a)$, $f(x+2a)$... is an A.P. What are " r " and " d " in these sequences ?

ANNUITIES.

PRELIMINARY REVIEW.

1. The amount of \$500 in 2 years with interest at 5 per cent. per annum compounded annually is $\$500(1.05)^2$ or \$551.25. Here \$500 is called the *present value* or *cash value* of \$551.25 due in 2 years, *i.e.* \$500 *now* is equivalent to \$551.25 *two years in the future* if money is worth 5 per cent. The \$51.25 is the interest *on* the \$500 and *in* the \$551.25.

The amount of \$1 under the same conditions is $\$(1.05)^2$. Hence the *future value* of \$500 cash is $500 \times \text{amt. of } \$1$. This illustrates the following general rule: To find the future sum that corresponds to a given cash sum, multiply by the amount of \$1.

2. The inverse of the above rule is obvious: To find the cash (present) sum that corresponds to a given future sum, *divide* by the amount of \$1.

Example: Find the cash value of \$1000 due in 2 years if interest is at 5 per cent. per annum, compounded yearly.

Required cash value $= \frac{1000}{(1.05)^2} = \$907.03$. The interest is $\$1000 - \$907.03 = \$92.97$. This is the interest *on* \$907.03 and *in* \$1000.

This \$1000 is spoken of sometimes as “\$1000 due in 2 years *without interest*, money being worth 5 per cent.,” the assumption being that the interest is included *in* the \$1000 so that no interest is to be computed *on* the \$1000.

Note that $\frac{1000}{(1.05)^2} = 1000 \times \frac{1}{(1.05)^2} = 1000 \times \text{present value of } \$1 \text{ due in 2 years}$. Hence the inverse rule may be stated as follows: To find the cash sum that corresponds to a given future sum, multiply by the cash value of a future dollar. The future equivalent of a present dollar and the cash equivalent of a future dollar will be found in the tables compiled for that purpose.

The process of finding the cash value of a future sum of money is called *discounting* the future sum. The difference between the two sums is called the *discount* on the future sum. In the last example \$1000 is the future sum, \$907.03 is the present sum, while \$92.97 is the discount on the \$1000. The discount on the future sum is the interest on the present sum. It should be noted that this discount, sometimes called the true discount, is not to be confused with the *bank* discount, which is interest reckoned *on the future sum*.

An ANNUITY is a succession of equal sums of money due at equal intervals of time. The intervals of time are usually a year in length : hence the word *annuity*.

CASH VALUE OF AN ANNUITY.

Examples.

1. Find the cash value of an annuity of \$1000 (per year) beginning now, and having 20 years to run, money worth 5 per cent.

We have to discount 20 payments of \$1000 each. The first payment is due one year from to-day. Its cash value is $\frac{1000}{1.05}$. The second payment is due 2 years from to-day. Its cash value is $\frac{1000}{(1.05)^2}$: and so on. Hence the cash value of the annuity is

$$\frac{1000}{1.05} + \frac{1000}{(1.05)^2} + \frac{1000}{(1.05)^3} + \dots$$

to 20 terms. This is a geometric series with a common ratio $\frac{1}{1.05}$. Hence the cash value of the annuity is

$$\begin{aligned} & \frac{\frac{1000}{1.05} \left[1 - \frac{1}{(1.05)^{20}} \right]}{1 - \frac{1}{1.05}} = \frac{\frac{1000}{1.05} \left[1 - \frac{1}{(1.05)^{20}} \right]}{\frac{0.05}{1.05}} \\ & = 20,000 \left[1 - \frac{1}{1.05^{20}} \right] = 20,000 \times 0.62311 = \$12,462.20. \end{aligned}$$

2. Find the cash value of an annuity of \$100 *deferred* 5 years and having 20 years to run, money worth 5 per cent.

Since the annuity is deferred 5 years the first payment is due in 6 years. The cash value of the annuity is

$$\begin{aligned}
 & \frac{100}{(1.05)^6} + \frac{100}{(1.05)^7} + \frac{100}{(1.05)^8} + \dots \text{to 20 terms} \\
 &= \frac{\frac{100}{(1.05)^6} \left[1 - \frac{1}{(1.05)^{20}} \right]}{1 - \frac{1}{1.05}} = \frac{2000}{(1.05)^5} \left[1 - \frac{1}{(1.05)^{20}} \right] \\
 &= 2000 \left[\frac{1}{(1.05)^5} - \frac{1}{(1.05)^{25}} \right] = 2000[0.78353 - 0.29530] \\
 &= 2000 \times 0.48823 = \$976.46.
 \end{aligned}$$

Annuities are frequently involved in financial documents which do not explicitly say so.

3. Find the cash value of a \$100 bond having 10 years to run, bearing interest at 6 per cent. and redeemable at par, money being worth 5 per cent.

Cash value of bond redemption

$$= \frac{100}{(1.05)^{10}} = 100 \times 0.61391 = \$61.391.$$

The interest payments constitute an annuity of \$6 per annum for 10 years. Hence the cash value of these

interest payments is $\frac{6}{1.05} + \frac{6}{(1.05)^2} + \frac{6}{(1.05)^3} + \dots$ to

10 terms = \$46.33. Therefore the cash value of the bond is \$61.39 + 46.33 = \$107.72.

ACCUMULATED VALUE OF AN ANNUITY LEFT ON DEPOSIT.

Examples.

1. An annuity of \$1000, beginning now, has 15 years to run. It is left uncollected by the payee, and accumulates at 4 per cent. Find its accumulated value at maturity.

The first payment is due at the end of the first year. It does not belong to the payee until that date, and therefore draws no interest for him until that date. *From then on* it belongs to the payee, and therefore the interest on it must be credited to him. *From the standpoint of the payee*, therefore, this instalment draws interest for 14 years. This instalment will amount to $\$1000(1.04)^{14}$ at the end of the 15 year period. Similarly, the second will amount to $\$1000(1.04)^{13}$, the third $\$1000(1.04)^{12}$, and so on. The last payment is due at the end of the 15 years, and draws no interest. The accumulated value of the annuity is

$$1000(1.04)^{14} + 1000(1.04)^{13} + 1000(1.04)^{12} + + \dots$$

to 15 terms. Verify the fact that the last term of this series is 1000. In practice it is preferable to add this series backwards. Thus the accumulated value of the annuity at the end of the 15 years is

$$1000 + 1000(1.04) + 1000(1.04)^2 + + \dots \text{to 15 terms}$$

$$= \frac{1000[(1.04)^{15} - 1]}{1.04 - 1} = \$20,023.50.$$

As an exercise the student should discount this sum for 15 years. He should also find the cash value of the annuity by the method shown in the preceding section. The two results should be the same. Why?

2. A manufacturer sets aside \$10,000.00 a year to meet depreciation in the value of machinery, making the first payment into this *depreciation fund* now. If the fund is invested at 5 per cent., find its accumulated value immediately after the tenth deposit.

$$\text{Accumulation} = 10,000 + 10,000(1.05) + 10,000(1.05)^2 + \dots \text{to 10 terms.}$$

$$= \frac{10,000[(1.05)^{10} - 1]}{1.05 - 1} = \$125,778.00.$$

Finding the Annual Payment.

1. What annuity beginning now (first payment due one year from now) will accumulate to \$10,000 in 12 years at 4 per cent. ?

Accumulated value of an annuity of

$$\begin{aligned} \$1 &= 1 + 1.04 + (1.04)^2 + \\ &+ \dots \text{to 12 terms} = \frac{(1.04)^{12} - 1}{1.04 - 1} = \frac{60.103}{4}. \end{aligned}$$

Hence the required annual payment is

$$\frac{\frac{10,000}{60.103}}{4} = 10,000 \times \frac{4}{60.103} = \$665.52.$$

2. What annuity beginning now and having 20 years to run will have a cash value of \$12,462.20 when discounted at 5 per cent. ?

Cash value of an annuity of \$1

$$\begin{aligned} &= \frac{1}{1.05} + \frac{1}{(1.05)^2} + \frac{1}{(1.05)^3} + \dots \text{to 20 terms} \\ &= \$12.4622. \end{aligned} \quad \text{Hence the required annual payment is}$$

$$\frac{12,462.20}{12.4622} = \$1000.$$

PERPETUITIES.

A PERPETUITY is an annuity that has an unlimited time to run.

Example : Discount at ⁵4 per cent. a perpetuity of \$100 beginning now. The cash values of the successive payments are

$$\frac{100}{1.05}, \quad \frac{100}{(1.05)^2}, \quad \frac{100}{(1.05)^3}, \quad \dots$$

The cash value of the perpetuity is *defined* as the limit of the sum of these payments when the number of payments is unlimited. This limit exists, for the common ratio, $\frac{1}{1.05}$, lies between -1 and $+1$.

Hence the cash value of the perpetuity is

$$\frac{\frac{100}{1.05}}{1 - \frac{1}{1.05}} = \$2000.$$

CAPITALISATION.

When an annual income or outgo that is regarded as a perpetuity is discounted, it is said to be *capitalised*. Such a process is called capitalisation.

For example, if the net income from land is assumed to be \$2.50 per acre per year in perpetuity, and we wish to capitalise the land at 5 per cent., we discount a perpetuity of \$2.50 at 5 per cent. Hence the capitalised (cash) value of the land *on the stated assumption* is the limit of the sum of the series $\frac{2.50}{1.05} + \frac{2.50}{(1.05)^2} + \frac{2.50}{(1.05)^3} + \dots$ when the number of terms is unlimited. This limit is 50. Hence the land is *capitalised* at \$50 per acre.

EXERCISE VI. f.

1. Find the cash value of an annuity of \$500 having 10 years to run, money worth 6 per cent.

2. Find the cash value of an annuity of \$100, deferred 5 years and having 20 years to run, money worth 5 per cent.

3. Find the accumulated value at maturity of an annuity of \$600 beginning now and having 15 years to run at 4 per cent. *Ans. 11,985.*

4. Discount a perpetuity of \$500 at $4\frac{1}{2}$ per cent.

5. What annuity to run 15 years can be bought for \$1037.96 cash if discounted at 5 per cent. ?

6. What annuity will amount to \$2977.80 in 20 years if left on deposit at 4 per cent. ?

7. A 10-year bond of \$100 bears interest at 5 per cent. If the bond is redeemable at par, discount it at 6 per cent.

8. A 20-year bond of \$100 bears interest at 6 per cent. If the bond sells at 98, what rate of interest does the investor receive ? *Q does not acc. @ 6%? Ans in book gives accum. value.*

9. A mortgage of \$10,000, bearing interest at 7 per cent., has 12 years to run. Discount the paper at 5 per cent.

10. If a man's expectation of life is 20 years, what will a life annuity cost him at $4\frac{1}{2}$ per cent. ?

11. Assuming the net income from a piece of land to be \$500 a year in perpetuity, capitalize the land at 5 per cent.

12. Find the cost of establishing a permanent endowment of \$10,000 at 4 per cent.

13. A city borrows \$500,000 cash at 6 per cent. to be repaid at the end of 15 years. Find the yearly sum required to pay the interest and to provide a *sinking fund* which, if accumulated at 5 per cent. will retire the debt at maturity.

14. The sum of \$50,000 is borrowed to be repaid in 10 equal annual payments covering both principal and interest at $5\frac{1}{2}$ per cent. Find the annual payment.

15. A father wishes to invest a yearly sum sufficient to provide his son with a 10-year annuity of \$1000. He makes the first investment on the son's tenth birthday, and wishes him to receive his first payment on his twenty-first birthday. Assuming that the fund accumulates at $4\frac{1}{2}$ per cent., and that the annuity is discounted at $5\frac{1}{2}$ per cent., find the yearly investment.

16. On a man's death it is found that according to the will the widow receives an annuity of \$1000 for life, and that on her death the annuity passes to the son. If her expectation of life is 20 years and the son's is 40 years, discount his annuity at 6 per cent.

\$3576.33

CHAPTER VII.

PERMUTATIONS AND COMBINATIONS.

COMPOUND EVENTS.

Example I. In an election campaign 5 names are on the ballot and an elector may vote for any of them. In how many ways can a vote by two electors X and Y be cast ?

The "compound event" consists of 2 votes, 1 by X and 1 by Y. The problem is to find the number of ways in which this compound event can take place.

Let us examine the ballot papers placed in the hands of X and Y.

Adams Baldwin Cooper Drummond Etter

Adams Baldwin Cooper Drummond Etter

If X votes for Adams, Y may vote for either Adams, Baldwin, Cooper, Drummond or Etter, giving 5 double votes. These double votes are Adams and Adams, Adams and Baldwin, Adams and Cooper, Adams and Drummond, Adams and Etter.

Similarly, if X votes for Baldwin, Y may vote for any one of the 5 names on the ballot, again giving 5 possible double votes. These are Baldwin and Adams, Baldwin and Baldwin, Baldwin and Cooper, Baldwin and Drummond, Baldwin and Etter.

Obviously, similar discussions apply when X votes for Cooper, when X votes for Drummond, and when X votes for Etter. Hence the total number of possible double votes is $5 \times 5 = 25$.

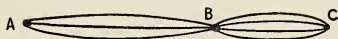
Using the initials of the names on the ballot, the 25 possible double votes may be exhibited as follows :—

AA	BA	CA	DA	EA
AB	BB	CB	DB	EB
AC	BC	CC	DC	EC
AD	BD	CD	DD	ED
AE	BE	CE	DE	EE

Obviously, this discussion of double votes may be abbreviated as follows :—

X has 5 choices of vote, *and for each of these* Y has 5 choices. Hence the total number of double votes is $5 \times 5 = 25$; or Y has 5 choices of vote *and for each of these* X has 5 choices. Hence the total number of double votes is $5 \times 5 = 25$.

Example II. Three railways run from A to B and 4 from B to C. How many kinds of through tickets must be printed to provide for all possible routings from A to C ?



A passenger may be routed from A to B in 3 ways, *and for each of these* he may be routed from B to C in 4 ways. Hence the total number of kinds of through tickets is $3 \times 4 = 12$.

The students should now discuss the following questions :
 (1) How many kinds of through tickets must be printed to provide for all possible routings from C to A ? (2) How

many kinds of return tickets must be printed to provide for all possible routings from A to C and return ? (3) How many kinds of return tickets must be printed to provide for all possible routings from C to A and return ?

PERMUTATIONS.

Example III. In how many ways can 3 things be arranged in a row ?

Let us label the things a, b, c . There are 3 places (), (), () to be filled by these 3 things. The first place can be filled in 3 ways, *viz.*, with either a, b , or c , and for each of these the second place can be filled in 2 ways. Thus if a occupies first place, the second place may be occupied by either b or c ; if b occupies first place the second may be occupied by either a or c , while if c occupies first place the second place may be occupied by either a or b . Thus the total number of ways of filling the first two places is $3 \times 2 = 6$. For each of these six ways of filling the first two places there is one way of filling the third place. For example, if the first two places are occupied by $(b), (c)$ the third place can be occupied by a only. The total number of ways of filling the 3 places is $3 \times 2 \times 1 = 6$.

The actual arrangements may be exhibited as follows :

abc	bac	cab
acb	bca	cba

We discussed the 3 places in their physical order. There is no necessity in this. Thus the discussion could proceed as follows : There are 3 ways of filling the third place, and for each of these there are 2 ways of filling the first place, giving a total of $3 \times 2 = 6$ ways of filling these two places. For each of these there is one way of filling the remaining place. Hence the total number of ways of filling the 3 places is $3 \times 2 \times 1 = 6$.

The student should practice carrying through the argument by taking the places in various orders. It should be noted that each place can be filled in 3 different ways.

The product $3 \times 2 \times 1$ is frequently represented by the symbol $3!$, which is called "factorial 3."

Arrangements are called PERMUTATIONS. The number of permutations of 3 things, then, is $3!$, or 6.

Example IV. In how many ways can n things be arranged in a row?

The argument, following the method in example I, may be carried through as follows: the first place can be filled in n different ways, and for each of these the second can be filled in $n - 1$ ways, making $n(n - 1)$ ways of filling these two places. For each of these ways of filling the first two places the third can be filled in $n - 2$ ways, making $n(n - 1)(n - 2)$ ways of filling the first 3 places, and so on. Hence the number of permutations is

$$n(n - 1)(n - 2)(n - 3) \dots 4 \cdot 3 \cdot 2 \cdot 1 \text{ or } n!$$

Example V. The number of ways of arranging 6 things is $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

Example VI. In how many different ways can 3 places be filled when we have 5 things at our disposal?

Suppose the things are labelled a, b, c, d, e . One way of filling the 3 places is $(b)(d)(c)$. We now proceed to the argument, taking the places in their physical order for simplicity.

The first place can be filled in 5 different ways, and for each of these the second can be filled in 4 ways, giving 5×4 ways of filling these two places. For each of these ways of filling these two places the third place can be filled in 3 ways, giving $5 \times 4 \times 3 = 60$ ways of filling the 3 places.

Obviously this problem can be stated as follows : How many permutations, each containing 3 things, can be made when we have 5 things at our disposal ? The problem is usually stated as follows : Find the number of permutations of 5 things taken 3 at a time.

The number of such permutations is represented by the symbol ${}^5P_3$. Hence ${}^5P_3 = 5 \times 4 \times 3 = 60$.

Note that the symbol ${}^5P_5$ represents the number of permutations of 5 things taken 5 at a time. This, of course, is simply the number of ways of arranging 5 things in a row. This number is $5! = 120$.

Example VII. Define and evaluate the symbol ${}^nP_r$ where n and r are positive integers and $n \geq r$.

Consistently with the preceding discussion we define the symbol ${}^nP_r$ as the number of permutations of n things taken r at a time.

If we repeat the argument of example VI, replacing 5 by n and 3 by r we get

$$\begin{aligned} {}^nP_r &= n(n-1)(n-2)(n-3) \dots \text{to } r \text{ factors} \\ &= n(n-1)(n-2)(n-3) \dots (n-r+2)(n-r+1). \end{aligned}$$

Example VIII. Find the number of permutations of 10 things taken 4 at a time.

We may solve the problem either by repeating an argument like that used in example VI or by using the formula for ${}^nP_r$.

Substituting in the formula we get

$${}^{10}P_4 = 10 \times 9 \times 8 \times 7 = 5040.$$

The student should also solve the problem by repeating the argument.

Example IX. ${}^nP_n = n!$

EXERCISE VII. a.

1. State the meanings of the following symbols and evaluate them twice, once by a discussion of "first principles" and once by substituting in the formula :

$${}^5P_2, {}^7P_3, {}^{10}P_4, {}^{10}P_5, {}^6P_6, {}^7P_7.$$

2. In how many ways can the product $abcde$ be written ?

3. In how many ways can 8 books be arranged on a shelf ?

4. There are 10 seats in a boat. In how many ways can 6 passengers be placed ? 10 passengers ?

5. Find the following quotients :

$$\frac{12!}{11!}, \frac{12!}{10!}, \frac{1000!}{999!}, \frac{1000!}{998!}, \frac{n!}{n-1!}, \frac{n!}{n-2!}, \frac{n!}{n-3!}, \frac{2n!}{n!}$$

6. If a colour scheme is a signal, how many signals can be made by hoisting 4 flags in a column, given 1 flag of each of 9 different colours ? Given at least 4 flags of each of 9 different colours ?

7. Solve the equation ${}^nP_2 = 3\frac{1}{2} \cdot {}^{n-3}P_2$.

8. How many positive integers of 4 digits can be written down if digits may be repeated ? If repetitions are excluded ? Will you solve by formula ?

9. How many "changes" can be rung with a peal of 8 bells ?

10. How many permutations can be made by using all the letters of the word *education* ? How many of these begin with e ? How many end with n ? How many begin with e and end with n ?

11. 5 ladies and 5 gentlemen are to be seated at a round table, the ladies and gentlemen to occupy alternate places. How many such arrangements are possible if we consider only the mutual positions of the guests ? How many arrangements are possible if we include position relative to the table ?

12. In how many ways can 6 boys stand in line ? In how many ways if 2 specified boys are not to be together ? In how many ways if a specified boy is not to be at either end of the line ?

COMBINATIONS.

Example X. In how many ways can 2 things be chosen from 5 things ?

Let us label the things a, b, c, d, e . The actual selections are $ab, ac, ad, ae, bc, bd, be, cd, ce, de$. There are 10 of these choices or selections.

Note that such *selection* can give rise to 2 *permutations*. Thus the *selection* ab can be *arranged* in 2 ways, ab and ba . These are two different permutations, *but they are the same selection*. The same holds true for the other selections. We should expect, then, that the number of selections would be one-half the number of permutations of 5 things taken 2 at a time. Now ${}^5P_2 = 5 \times 4 = 20$, while, as we have seen, the number of selections is 10. This verifies our expectation.

Each *choice* or *selection* is called a COMBINATION.

Example XI. How many combinations can be made, each containing 3 things, when we have 6 things at our disposal ? In other words, in how many ways can 3 things be chosen from 6 things ?

Let the things be labelled a, b, c, d, e, f . The *selections* or *choices* or *combinations* may be exhibited as follows :

abc	acd	ade	$ae f$
abd	ace	adf	
abe	acf		
abf			
bcd	bde	bef	
bce	bdf		
bcf			
cde	cef		
cdf			

def , a total of 20 combinations.

Note that the combination abc can give rise to $3!$ or 6 permutations, *viz.*, $abc, acb, bac, bca, cab, cba$. These are 6 different permutations, *but they are one and the same combination*. Similarly, each of the other combinations can give rise to 6 permutations of 3 things. We should

expect, then, that the number of combinations would be $\frac{1}{6}$ the number of permutations. Now ${}^6P_3 = 6 \times 5 \times 4 = 120$, while, as we have seen, the number of combinations is 20. Our expectation is verified.

Note that COMBINATION refers merely to the things chosen *without regard to their order*, while PERMUTATION refers *both to the things chosen and their order*.

The above examples illustrate the fact that *there is a definite relation between the number of combinations and the number of corresponding permutations*.

Example XII. How many combinations, each containing 9 things, are possible when we have 50 things from which to choose?

Each combination of 9 things can give rise to $9!$ permutations. Hence the number of combinations is $\frac{1}{9!}$ the number of permutations of 50 things taken 9 at a time. But the number of these permutations is ${}^{50}P_9$. Hence the required number of combinations is $\frac{{}^{50}P_9}{9!}$.

Usually, instead of saying "the number of combinations, each containing 9 things, when there are 50 things from which to choose," we say "the number of combinations of 50 things taken 9 at a time."

The number of combinations of 50 things taken 9 at a time is represented by the symbol ${}^{50}C_9$. Hence ${}^{50}C_9 = \frac{{}^{50}P_9}{9!}$.

The number of combinations of n things taken r at a time is represented by the symbol ${}^nC_r$.

Example XIII. Evaluate the symbol ${}^nC_r$.

Each combination of r things can give rise to $r!$ permutations.

$$\therefore {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n(n-1)(n-2) \dots (n-r+2)(n-r+1)}{r!}.$$

Example XIV. In how many ways can 4 things be chosen from 9 things? (How otherwise can this question be worded?)

$${}^9C_4 = \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 126.$$

Another formula for ${}^nC_r$:

We have

$${}^nC_r = \frac{n(n-1)(n-2) \dots (n-r+2)(n-r+1)}{r!}.$$

Multiplying both numerator and denominator by $n-r!$ we get

$$\begin{aligned} {}^nC_r &= \frac{n(n-1)(n-2) \dots (n-r+2)(n-r+1)(n-r)(n-r-1) \dots 3 \cdot 2 \cdot 1}{-r! \quad n-r!} \\ &= \frac{n!}{r! \quad n-r!} \end{aligned}$$

Theorem :

$${}^nC_r = {}^nC_{n-r}.$$

Proof :

$$\begin{aligned} {}^nC_r &= \frac{n!}{r! \quad n-r!} \\ {}^nC_{n-r} &= \frac{n!}{n-r! \quad n-(n-r)!} = \frac{n!}{n-r! \quad r!} \\ \therefore {}^nC_r &= {}^nC_{n-r}. \end{aligned}$$

For example, ${}^5C_3 = {}^5C_2$. The student should verify this by formula and by physical test.

Example XV. In how many ways can a committee of 3 be chosen from 10 men?

$${}^{10}C_3 = \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = 120.$$

Note that the number of ways of choosing the 3 men and arranging them in a row is $120 \cdot 3! = 720$, or

$${}^{10}P_3 = 10 \cdot 9 \cdot 8 = 720.$$

Example XVI. In how many ways can a committee of 2 physicians and 5 laymen be chosen from 7 physicians and 15 laymen?

The physicians can be chosen in ${}^7C_2$ ways, and for each of these the laymen can be chosen in ${}^{15}C_5$ ways. Hence the number of ways of choosing the committee is

$$\begin{aligned} {}^7C_2 \times {}^{15}C_5 &= \frac{7 \cdot 6}{1 \cdot 2} \times \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \\ &= 63,063. \end{aligned}$$

EXERCISE VII. b.

1. In how many ways can 2 men be chosen from 5 men? What is the value of ${}^5C_2$?

2. In how many ways can a football eleven be chosen from 14 men? What is the value of ${}^{14}C_{11}$? Of ${}^{14}C_3$?

3. In how many ways can 3 cards be chosen from a pack of 52 cards?

4. How many hands is it possible for a player to have at bridge? Evaluate the number to two significant figures by logarithms.

5. There are 12 points in a plane, no 3 of which are collinear.

(1) How many straight lines can be drawn joining pairs of points?

(2) How many triangles can be drawn each having 3 of these points as vertices?

6. In how many ways can a committee of 4 men and 2 women be appointed from a society of 12 men and 8 women?

7. (1) In how many ways can 2 teams of 6 players each be chosen from 12 men? (2) In how many ways can 3 teams of 4 players each be chosen?

8. From 4 teachers and 12 pupils in how many ways can a committee of 5 be chosen to include 1, and only 1, teacher? To include *at least* 1 teacher?

9. Find the number of combinations of 100 things taken 97 at a time.

✓ 10. What is the meaning of the symbol ${}^nC_n$? Its value ?

11. Find *meanings* for ${}^nP_0$, ${}^nC_0$, $0!$ consistent with the *laws* governing ${}^nP_r$, ${}^nC_r$, $r!$ —

✓ 12. In how many ways can 6 different books be distributed among A , B and C , so that A has one book, B two, and C has three ?

CHAPTER VIII.

THE BINOMIAL THEOREM.

IF we multiply out $(x + a)(x + b)$ we obtain various terms, each of which has two factors, one from the first bracket and the other from the second bracket.

Thus $(x + a)(x + b) \equiv x^2 + ax + bx + ab$.

In the same way, if we multiply out $(x+a)(x+b)(x+c)$ we get various terms, each of which has three factors, one from each bracket. These terms can be conveniently grouped according to the number of x factors chosen in each term.

If the x is chosen from each bracket, we get the term x^3 .

If the x is chosen from two brackets and the remaining letter from the third bracket, we get ${}^3C_1$ terms, namely the terms $ax^2 + bx^2 + cx^2$.

If the x is chosen from one bracket and the remaining letter from the other two brackets, we get ${}^3C_2$ terms (the number of ways in which the other two letters can be chosen), and these terms are $abx + bcx + cax$.

Finally, if no x term is chosen, we get one term abc .

Thus $(x + a)(x + b)(x + c)$ can be written down by inspection as

$$x^3 + x^2(a + b + c) + x(ab + bc + ca) + abc.$$

Note that if we now make b and c equal to a , we find that

$$\begin{aligned} (x+a)(x+a)(x+a) &\equiv x^3 + x^2(a+a+a) + x(a^2+a^2+a^2) + a^3 \\ &\equiv x^3 + 3ax^2 + 3a^2x + a^3. \end{aligned}$$

EXERCISE VIII. a.

1. Write down the expansions of

~~(i)~~ $(x + a)(x + b)(x + c)(x + d)$;

(ii) $(x + a)(x + a)(x + a)(x + a)$;

(iii) $(x + 1)(x + 1)(x + 1)(x + 1)$;

(iv) $(x - 1)^4$.

2. (i) How many terms are there in the expansion of $(x + a)(x + b)(x + c)(x + d)(x + e)(x + f)$?

(6) (ii) How many of these terms have a factor x and not x^2 ?

(15) (iii) How many of these terms have x^2 as a factor and not x^3 ?

3. What is the coefficient of x^2 in the expansions of

(i) $(x + a)(x + b)(x + c)(x + d)(x + e)$;

~~(ii)~~ $(x + a)^5$;

(iii) $(x - 1)^5$?

4. How many terms in the following expansions contain x^3 but not x^4 as a factor :

(i) $(x + a)(x + b)(x + c)(x + d)$;

(ii) $(x + a_1)(x + a_2)(x + a_3)(x + a_4)(x + a_5)(x + a_6)(x + a_7)$?

What is the coefficient of x^3 in $(x + 1)^7$?

5. Expand in descending powers of x :

(i) $(x + 1)(x + 2)(x + 3)$;

(ii) $(x + 1)(x + 2)(x + 3)(x + 4)$.

6. Expand in descending powers of x :

~~(i)~~ $(x + 1)(2x + 1)(3x + 1)$;

(ii) $(x + 1)(2x + 1)(3x + 1)(4x + 1)$.

The examples of the last exercise show how to write down the expansion of any expression of the form

$$(x + a_1)(x + a_2)(x + a_3)(x + a_4) \dots (x + a_n).$$

It is, in fact,

$$x^n + x^{n-1}(a_1 + a_2 + a_3 + \dots) + x^{n-2}(a_1a_2 + a_1a_3 + a_2a_3 + \dots) + x^{n-3}(a_1a_2a_3 + a_1a_2a_4 + \dots) + \dots$$

The coefficient of x^{n-2} is the sum of the products, taken two at a time, of the n letters $a_1, a_2, \dots, a_n$.

The coefficient of x^{n-3} is the sum of these products taken three at a time.

The coefficient of x^{n-4} is the sum of these products taken *four* at a time, and so on.

Therefore, there are n terms in the coefficient of x^{n-1} ,
 and ${}^nC_2 \dots x^{n-2}$,
 and ${}^nC_3 \dots x^{n-3}$,
 and so on.

Suppose now $a_1 = a_2 = a_3 = a_4 = \dots = a_n = a$.

Then each term in the coefficient of x^{n-1} equals a^2 ,
 and $\dots x^{n-2} \dots a^2$,
 and $\dots x^{n-3} \dots a^3$,
 and so on.

$\therefore$ the coefficient of x^{n-1} is $n \cdot a$,
 $\dots x^{n-2}$ is ${}^nC_2 \cdot a^2$,
 $\dots x^{n-3}$ is ${}^nC_3 \cdot a^3$,

and so on.

$\therefore$ if n is any positive integer,

$$(x+a)^n \equiv x^n + n \cdot x^{n-1}a + {}^nC_2 \cdot x^{n-2}a^2 + {}^nC_3 \cdot x^{n-3}a^3 + \dots + {}^nC_r x^{n-r}a^r + \dots + a^n,$$

or

$$(x+a)^n \equiv x^n + n \cdot x^{n-1}a + \frac{n(n-1)}{1 \cdot 2} x^{n-2}a^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^{n-3}a^3 + \dots + \frac{n(n-1)(n-2) \dots (n-r+1)}{1 \cdot 2 \cdot 3 \dots r} x^{n-r}a^r + \dots + a^n.$$

This result is called the *Binomial Theorem*.

EXERCISE VIII. b.

Write down the expansions of Nos. 1-8.

1. $(1+x)^3$. 2. $(1+x)^4$. 3. $(1-x)^4$. 4. $(x-2)^3$.

5. $(x-y)^6$. 6. $\left(x + \frac{1}{x}\right)^5$. 7. $(2x-3y)^3$. 8. $(a^2-2b)^4$.

9. What is the coefficient x^3 in (i) $(x-1)^7$; (ii) $(1+2x)^n$?

10. What is the 4th term in the expansion of $(2-x)^{10}$?

11. What is the term independent of x in the expansion of $\left(x + \frac{2}{x}\right)^4$?
12. How many terms are there in the expansion of $(2x - 3y)^{10}$?
13. What is the middle term in the expansion of $(x - y)^6$?
14. What is (i) the 4th term from the beginning; (ii) the 4th term from the end in $(x + y)^{10}$?
15. What is the r th term in the expansion of (i) $(1 + 2x)^n$; (ii) $(1 - 2x)^n$?
16. What is (i) the r th term from the beginning; (ii) the r th term from the end in $(x + y)^n$? Are the coefficients equal?
17. Simplify $(x - 1)^3 + 3(x - 1)^2 + 3(x - 1) + 1$.
18. Simplify $1 - 4(x + 1) + 6(x + 1)^2 - 4(x + 1)^3 + (x + 1)^4$.
19. In the identity $(1 + x)^n \equiv 1 + {}^nC_1 \cdot x + {}^nC_2 \cdot x^2 + \dots + {}^nC_r \cdot x^r + \dots + x^n$, what results are obtained by putting (i) $x = 1$; (ii) $x = -1$?
20. What is the coefficient of x^4 in $(1 - x)(1 + x)^7$?
21. What is the coefficient of x^r in $(1 - x)(1 + x)^n$?
22. Expand $(1 + x + x^2)^n$ as far as terms involving x^3 .
23. Simplify $(\sqrt{2} + 1)^4 + (\sqrt{2} - 1)^4$.
24. Use the binomial theorem to evaluate 101^4 and 99^5 .
25. Use the binomial theorem to evaluate to 5 significant figures 1.02^5 and 0.97^6 .

APPROXIMATION.

The binomial expansion

$$(1 + x)^n \equiv 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \dots$$

$$+ \frac{n(n-1)(n-2) \dots (n-r+1)}{1 \cdot 2 \cdot 3 \dots r}x^r + \dots$$

has been proved, if n is a positive integer.

If n is fractional or negative, this result still holds good, provided that x lies between -1 and $+1$. The general proof is, however, beyond the scope of this book.

If n is a positive integer, the expansion contains $n + 1$ terms.

But if n is fractional or negative, the expansion in this form does not terminate, *i.e.* is an infinite series; in this case, if x and nx are small compared with 1, the first few terms of the expansion may give a useful approximation.

EXERCISE VIII. c.

1. Prove by long division that

$$(i) \quad \frac{1}{1-x} \equiv 1 + x + \frac{x^2}{1-x};$$

$$(ii) \quad \frac{1}{1-x} \equiv 1 + x + x^2 + \frac{x^3}{1-x};$$

$$(iii) \quad \frac{1}{1-x} \equiv 1 + x + x^2 + x^3 + \frac{x^4}{1-x}.$$

2. Find R if $\frac{1}{1-x} \equiv 1 + x + x^2 + x^3 + x^4 + R$.

3. Assuming the binomial theorem, if x is a positive fraction less than 1, write down the first five terms of the expansion of

$$\frac{1}{1-x} \equiv (1-x)^{-1}.$$

4. If $x = 0.1$, what is approximately the error in taking $\frac{1}{1-x}$ equal to (i) $1 + x$; (ii) $1 + x + x^2$?

5. Find R by long division if

$$(i) \quad \frac{1}{1+x} \equiv 1 - x + R;$$

$$(ii) \quad \frac{1}{1+x} = 1 - x + x^2 + R;$$

$$(iii) \quad \frac{1}{1+x} = 1 - x + x^2 - x^3 + R.$$

6. Assuming the binomial theorem, if x is a positive fraction less than 1, write down the first five terms of the expansion of

$$\frac{1}{1+x} \equiv (1+x)^{-1}.$$

If $x = 0.01$, what is approximately the error in taking $\frac{1}{1+x}$ equal to (i) $1-x$; (ii) $1-x+x^2$?

7. Find R if (i) $\frac{1}{(1-x)^2} \equiv 1+2x+R$;

cor. (ii) $\frac{1}{(1+x)^2} = 1-2x+R.$

8. Assuming the binomial theorem, if x is a positive fraction less than 1, write down the first four terms of the expansions of

$$\frac{1}{(1-x)^2} \equiv (1-x)^{-2} \text{ and } \frac{1}{(1+x)^2} \equiv (1+x)^{-2}.$$

What is approximately the error in taking $(1-x)^{-2} = 1+2x$ and $(1+x)^{-2} = 1-2x$ if $x = 0.01$?

9. If x is a positive fraction less than 1, prove that

- (i) $(1-\frac{1}{2}x)^2 > 1-x$; (ii) $(1-\frac{1}{2}x-\frac{1}{2}x^2)^2 < 1-x$;
- (iii) $1-\frac{1}{2}x-\frac{1}{2}x^2 < \sqrt{1-x} < 1-\frac{1}{2}x$;
- (iv) the error in taking $\sqrt{1-x} = 1-\frac{1}{2}x$ is less than $\frac{1}{2}x^2$.

10. (i) Find by the square root rule the value of $\sqrt{1-x}$ as far as terms involving x^3 .

(ii) What are the first four terms of the binomial expansion for $\sqrt{1-x} \equiv (1-x)^{\frac{1}{2}}$, assuming the Binomial Theorem?

(iii) If $x = 0.01$, what is the approximate error in taking $\sqrt{1-x} = 1-\frac{1}{2}x$?

11. (i) Find by the square root rule the value of $\sqrt{1+x}$ as far as terms involving x^3 .

(ii) If x is a positive fraction less than 1, what are the first four terms of the binomial expansion for

$$\sqrt{1+x} \equiv (1+x)^{\frac{1}{2}},$$

assuming the Binomial Theorem?

(iii) If $x = 0.1$, what is the approximate error in taking $(1+x)^{\frac{1}{2}} = 1+\frac{1}{2}x$?

12. (i) Prove that $\frac{1}{x+h} \equiv \frac{1}{x} \left(1 + \frac{h}{x}\right)^{-1}$.

(ii) If h is small compared to x , find an approximation for

$$\frac{1}{x+h}.$$

(iii) Express approximately $\frac{1}{1.02}$ as a decimal.

13. (i) Prove that $\frac{1}{\sqrt{x^2+h}} \equiv \frac{1}{x} \left(1 + \frac{h}{x^2}\right)^{-\frac{1}{2}}$.

(ii) If h is small compared to x , find an approximation for

$$\frac{1}{\sqrt{x^2+h}}.$$

(iii) Express approximately $\frac{1}{\sqrt{4.01}}$ as a decimal.

(iv) Express approximately $\frac{1}{\sqrt{9.05}}$ as a decimal.

14. Criticize the following evaluations :—

(i) $(1.01)^2 = 1.0201, \quad 1.020, \quad 1 + 2(0.01).$

(ii) $(1.01)^3 = 1.030301, \quad 1.030, \quad 1 + 3(0.01).$

(iii) $(1.01)^4 = 1.04060401, \quad 1.041, \quad 1 + 4(0.01).$

(iv) $(1.01)^{100} = 2.705, \quad 1 + 100(0.01).$

(v) $(1.001)^{1000} = 2.717, \quad 1 + 1000(0.001).$

15. Discuss $(1+x)^n \simeq 1+nx$, when both x and nx are (i) small, (ii) large, compared with 1.

16. (i) List with examples the kinds of series found in Chapter VI. and this Chapter.

(ii) Give examples of series that belong to more than type.

SUMMARY OF RESULTS.

I. If x and nx are small compared with 1,

$$\begin{aligned}(1+x)^n &\simeq 1 + nx, \\ (1-x)^n &\simeq 1 - nx.\end{aligned}$$

In particular,

$$\begin{aligned}\frac{1}{1+x} &\equiv (1+x)^{-1} \simeq 1-x, \\ \frac{1}{1-x} &\equiv (1-x)^{-1} \simeq 1+x, \\ \sqrt{1+x} &\equiv (1+x)^{\frac{1}{2}} \simeq 1 + \frac{1}{2}x, \\ \sqrt{1-x} &\equiv (1-x)^{\frac{1}{2}} \simeq 1 - \frac{1}{2}x.\end{aligned}$$

II. If h is small compared with x ,

$$(x+h)^n \equiv x^n \left(1 + \frac{h}{x}\right)^n \simeq x^n \left(1 + \frac{nh}{x}\right).$$

Example I. The cost per hour of an electric lamp varies as $\frac{V^2}{R}$, where V is the voltage and R is the resistance. Find the approximate percentage change in the cost if the voltage is increased by 2 per cent. and the resistance is decreased by 3 per cent.

If the original cost per hour is $\frac{kV^2}{R}$ cents, the new cost is

$$\begin{aligned}\frac{k(V \times 1.02)^2}{R \times 0.97} \text{ cents} &= \frac{kV^2}{R} (1 + 0.02)^2 (1 - 0.03)^{-1} \text{ cents} \\ &\simeq \frac{kV^2}{R} (1 + 0.04)(1 + 0.03) \text{ cents} \\ &\simeq \frac{kV^2}{R} (1 + 0.04 + 0.03) \text{ or } \frac{1.07kV^2}{R} \text{ cents}; \\ \therefore \text{ the increase of cost is } &\frac{0.07kV^2}{R} \text{ cents}; \\ \therefore \text{ the increase is } &7 \text{ per cent.}\end{aligned}$$

EXERCISE VIII. d.

1. (i) Simplify $(1 + a)(1 + b) - (1 + a + b)$.
 (ii) What is the error in taking 1.01×1.02 equal to $1 + .01 + .02$ or 1.03 ?
 (iii) Write down the approximate value of 1.03×1.05 .
2. (i) If h is small, what is an approximate value of $(1 + h)^3$?
 (ii) Write down an approximate value of $(1.02)^3$.
3. Each linear dimension of a rectangular block of steel expands 0.000,011 of itself for every 1°C. rise of temperature. In what ratio does the volume increase if the temperature rises (i) 1°C. , (ii) 3°C. ?
4. If an error of 1 per cent. is made in measuring the length and breadth of a field, what may be the approximate error per cent. in the calculated area ?
5. A man estimates the dimensions of a room, and makes an error of 1 per cent. in the length, 2 per cent. in the breadth, and 3 per cent. in the height ; what may be the approximate error per cent. in the calculated volume ?
6. If the diameter of a sphere increases by 1 per cent., what is the approximate increase per cent. (a) in its volume, (b) in its surface ?

prob. implies perfect cube of steel.

7. If h is small,

(i) show that $\frac{1 - h}{1 + h} \approx 1 - 2h$;

(ii) find an approximation for $\frac{1 + h}{1 - h}$, and hence for $\frac{1.01}{0.99}$.

8. Find approximations for

(i) $(1.002)^5$; (ii) $(0.98)^3$; (iii) $\frac{1}{0.997}$;

(iv) $\sqrt{1.008}$; (v) $\sqrt[3]{1.015}$; (vi) $\frac{1}{5.003}$.

9. If h is small, find approximate values of

(i) $\frac{(1 + 3h)(1 + 4h)}{1 + 5h}$; (ii) $\frac{(1 + 3h)(1 + 4h)}{1 + 12h}$.

What is the approximate value of $\frac{1.03 \times 1.04}{1.12}$?

10. The time of oscillation of a pendulum of length l feet is approximately $1.11\sqrt{l}$ seconds. Find approximately the increase in the time when the length is altered from (i) 4 feet to 4.1 feet, (ii) l feet to $(l + h)$ feet, where h is small compared with l .

11. A given mass of gas at constant temperature obeys the law $pv = \text{constant}$, where p is the pressure in lb. per sq. inch and v is its volume in cu. inches.

- (i) If the pressure is increased by 2 per cent., what is the approximate change in the volume?
- (ii) If the pressure alters from p to $p + p_1$, what is approximately the change of volume if p_1 is small compared with p ?

12. If h is small compared with x , find approximate values for

$$\begin{aligned} \text{(i)} \quad & \frac{(x + h)^3 - x^3}{h}; & \text{(ii)} \quad & \frac{(x + h)^{10} - x^{10}}{h}; \\ \text{(iii)} \quad & \frac{1}{x + h} - \frac{1}{x}; & \text{(iv)} \quad & \frac{\sqrt{x + h} - \sqrt{x}}{h}. \end{aligned}$$

13. The maximum deflection of a beam of length l feet and depth d inches, supported at its ends, is $\frac{kl^4}{d^2}$ in., where k is a constant; in what ratio is the deflection altered if l increases from 20 to 20.1 and d increases from 4 to 4.1?

14. A given mass of gas, absolute temperature T° , volume v cu. inches at pressure p lb. per sq. inch, obeys the law $\frac{pv}{T} = \text{constant}$. Find the percentage change in p if v increases by 2 per cent. and T decreases by 3 per cent.

15. The time of pneumatic transmission through a pipe l yards long of diameter d inches under a pressure of p lb. per sq. inch varies as $\frac{l^3}{d^{\frac{5}{2}}p^{\frac{1}{2}}}$. What is the approximate percentage change in the time if l increases by 1 per cent., d increases by 2 per cent., and p decreases by 1 per cent.?

16. In the Michelson-Morley experiment for the determination of the velocity of the earth relative to the ether, the effect to be observed was measured by $2(l_1 + l_2) \left\{ \frac{c}{c^2 - v^2} - \frac{1}{\sqrt{c^2 - v^2}} \right\}$, where v is the velocity required and c the velocity of light. Prove that if $\frac{v}{c}$ is small, this expression $\simeq (l_1 + l_2) \cdot \frac{v^2}{c^3}$.

CHAPTER IX.

EMPIRICAL FORMULÆ.

AN experiment was made to determine the extension produced in a spiral spring by weights of various amounts attached to its extremity, and the results were tabulated as follows :—

Length of spring in cm., l	28.3	29.5	31.5	35.6	39.7
Weight attached in gr., W	10	25	50	100	150

These observed results can be shown to obey *approximately* the law $l = 0.081W + 27.5$.

A law obtained in this way is called an *empirical formula*.

If the result of plotting y against x gives a straight line, y and x are connected by an equation of the form $y = ax + b$, where a , b are constants. The following example shows how these constants are found, if the straight-line graph is given.

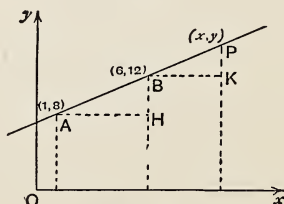


FIG. 50.

Example I. Two points on a straight-line graph which represents the relation between two variables x and y , are found to be $x = 1, y = 8$, and $x = 6, y = 12$.

Find the equation connecting x and y in the form

$$y = ax + b.$$

A, B are the points $(1, 8)$ and $(6, 12)$.

Denote the coordinates of *any other* point P on the line by (x, y) .

Through A, B, P draw lines parallel to Ox, Oy , forming the triangles AHB, BKP , which are similar.

$$\therefore \frac{KB}{HA} = \frac{PK}{BH};$$

$$\therefore \frac{x - 6}{6 - 1} = \frac{y - 12}{12 - 8}; \quad \therefore \frac{x - 6}{5} = \frac{y - 12}{4};$$

$$\therefore 4x - 24 = 5y - 60; \quad \therefore 5y = 4x + 36; \quad \therefore y = 0.8x + 7.2,$$

which is the functional relation required.

By exactly the same method, it may be shown that if (x_1, y_1) and (x_2, y_2) are the coordinates of any two points on a straight line graph, the relation which the graph represents is

$$\left[\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} \right] \text{ or } y = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Compare summary at end of Chapter III.

Example II. The observations made in an experiment were tabulated as follows :

$x = 1$	1.5	2	2.5	3	3.5	4
$y = 2.50$	2.20	1.92	1.65	1.32	0.94	0.60

Find the most probable relation which expresses y in terms of x .

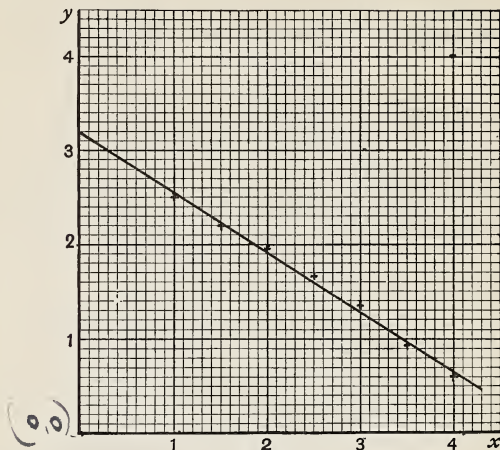


FIG. 51.

First plot the observations. We see that the points obtained, although not on a straight line, do not differ much from the straight. Now use a fine black thread to determine the "best-fit line" and read off the coordinates of two points at opposite ends: e.g., $x = 0.3$, $y = 3$, and $x = 3.9$, $y = 0.7$.

The relation is therefore

$$\frac{x - 3.9}{3.9 - 0.3} = \frac{y - 0.7}{0.7 - 3}$$

or

$$\frac{x - 3.9}{3.6} = \frac{y - 0.7}{-2.3};$$

$$\therefore y - 0.7 = -\frac{2.3}{3.6}(x - 3.9);$$

$$\therefore y = -\frac{2.3}{3.6}x + \frac{2.3 \times 3.9}{3.6} + 0.7;$$

$$\therefore y = -0.64x + 3.2,$$

where each constant is calculated correct to two significant figures.

Example III. A load of 100 lb. is fastened to one end of a rope which passes round a rough circular peg and measurements are made of the least pull which must be applied at the other end of the rope in order to raise the load slowly. The magnitude of the pull required depends on the amount of the rope in contact with the peg. Suppose the least pull is P lb. when the portion of the rope in contact with the peg is θ right angles. The following observations were made :

θ right angles	1	2	4	6	8
P lb. . . .	140	181	359	672	1190

Find the most probable relation expressing P in terms of θ .

If we plot these results as they stand, the graph will be a curve. If, however, we plot $\log P$ against θ we see that the points do not diverge far from a straight line. The table is

$\theta = 1$	2	4	6	8
$\log P = 2.1461$	2.2577	2.5551	2.8274	3.0755

These results are plotted in Fig. 52 : and a fine thread is then used to discover the "best-fit line." We see that two points on this line are

$$\begin{aligned}
 & \left. \begin{array}{l} \theta = 7.6 \\ \log P = 3.038 \end{array} \right\} \text{ and } \left. \begin{array}{l} \theta = 1.3 \\ \log P = 2.179 \end{array} \right\} ; \\
 & \therefore \text{ the relation is } \frac{\log P - 3.038}{3.038 - 2.179} = \frac{\theta - 7.6}{7.6 - 1.3} ; \\
 & \therefore \frac{\log P - 3.038}{0.859} = \frac{\theta - 7.6}{6.3} \quad \text{or} \quad \log P = \frac{0.859}{6.3} \\
 & \qquad \qquad \qquad \theta - \frac{0.859 \times 7.6}{6.3} + 3.038 ;
 \end{aligned}$$

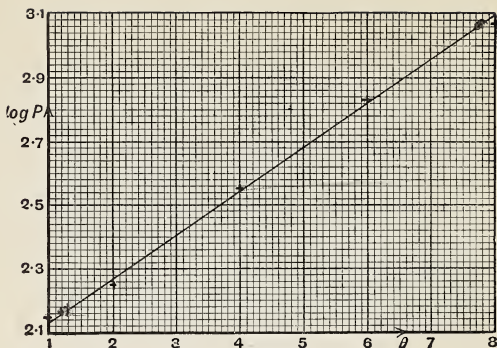


FIG. 52.

$$\therefore \log P = 0.136\theta + 2.002; \quad \therefore P = 10^{0.136\theta + 2.002};$$

$$\therefore P = 10^{2.002} \times 10^{0.136\theta}; \quad \therefore P = 100 \times 10^{0.136\theta}.$$

This approximate formula is the required relation.

EXERCISE IX. a.

1. The following observations were made :

$x = 1$	2	4	5	7
$y = 0.95$	2.8	5.8	7.7	10.7

Find the best fit equation for y in terms of x .

2. The length of a spiral spring is measured when different weights are suspended from its end ; the results, when tabulated, are as follows :

Weight in gr., W	10	15	30	50	75
Length in cm., l	22.1	24.1	30.2	38.4	48.7

Find the best fit equation for l in terms of W .

3. The following results were obtained with a screw jack :

Load in lb., W	10	20	30	50	75
Effort in lb., P -	3.28	4.51	5.72	8.11	11.12

Find the best fit equation for P in terms of W .

4. If $y = \frac{100}{\sqrt{100+x}}$, make a table showing corresponding values of x and y between $x = 10$ and $x = 50$, and find a relation of the form $y = a + bx$, which best fits this function in this range. What is the error when $x = 44$?

5. The following observations were made :

x	1	2	3	4	5
y	8	13	18	28.5	34

It is believed that they obey a linear law. In which observation is the error probably greatest ? Ignoring this, what is the best fit equation for y in terms of x ? And how would the table then run ?

6. The pressure P lb. per sq. foot exerted by a wind of v miles an hour is measured in the following cases :

v	5	10	15	20	25	30
P	0.12	0.51	1.15	1.88	2.95	4.58

Plot P against v^2 , and hence find the probable relation expressing P in terms of v .

7. In Giffard's Injector, the following values are given for the delivery of Q gallons of water per hour under a pressure of 60 lb. per sq. inch for a throat diameter D inches.

D	0.1	0.15	0.2	0.25	0.3
Q	300	730	1200	2000	2700

Plot D against $\sqrt{Q}$, and hence find the probable relation expressing Q in terms of D .

8. The following table gives the resistance R lb. to a train of weight 100 tons running at V miles an hour :

V	10	20	30	40	50	60
R	680	980	1380	2050	2880	3800

Plot R against V^2 , and then express R in terms of V .

9. A body of density d , weighing 1 gr. in air (when brass weights are used), weighs 1 gr. + k mgr. in a vacuum. The following observations were recorded :

$d = 0.5$	1	3	5	6
$k = 2.26$	1.06	0.26	0.10	0.06

Plot k against $\frac{1}{d}$, and express k in terms of d .

10. A marble rolls down a groove, at the side of which are markers which are so adjusted as to point the position of the marble at intervals of one second. The following observations were made :

Time in seconds, t	1	2	3	4	5
Distance rolled in inches, s	4.1	16.3	36.8	65.4	102.1

Find the probable relation expressing s in terms of t .

11. The following results were recorded in an optical experiment :

x	10	11	12	15	20	30
y	39.9	33.1	28.9	22.4	18.5	15.7

Plot $\frac{1}{x}$ against $\frac{1}{y}$, and find the best fit equation expressing y in terms of x .

12. The following list of British amateur running records is taken from *Whitaker's Almanack* :

Distance in yd., d	150	200	220	440	880	1000
Time in sec., t	14.6	19.4	21.8	48.4	112.2	132.4

Plot $\log t$ against $\log d$, and hence find a relation expressing t in terms of d .

Professor Perry has pointed out that all forms of race records (men and animals) conform to a law of the same kind.

13. In a radiation experiment, the following records were made :

Absolute temp., T	373.1	492.5	723	810	1378	1497
Deviation of galvanometer, δ	156	638	3320	5150	44700	61600

Plot δ against T^4 , and express δ in terms of T . [Stefan's Law.]

SUMMARY OF RESULTS.

(i) If the graph obtained by plotting y against x is, or approximates to, a straight line, and if (x_1, y_1) , (x_2, y_2) are two points on it, preferably at opposite ends, then y can be expressed as a function of x by the equation

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}, \text{ or } y = y_1 + \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

(ii) If the graph obtained by plotting y against x is a curve, it may be possible to obtain a straight line by plotting

- (a) $\log y$ against $\log x$,
- (b) $\log y$ against x ,
- (c) y against $\log x$,
- (d) y against $\frac{1}{x}$, etc.

(iii) Where the data are experimental, coefficients in the functional relation should be expressed as decimals, and must not be given to more significant figures than the data and the method justify.

REVISION EXERCISES.

A

1. Divide the cube of $2x^2$ by the square of $2x^3$.
2. How would the following facts be expressed without using the index notation ?
 - (i) the velocity of light is 3.002×10^{10} cm. per sec. ;
 - (ii) the population of London in 1920 was about 4.5×10^6 ;
 - (iii) the value of steel imports in 1913 was about $\pounds 12.2 \times 10^6$;
 - (iv) the distance of α Centauri is 2.6×10^{13} miles ;
 - (v) $2 \times 10^3 \times 4.5 \times 10^6 = 9 \times 10^9$.
3. By using tables, write down the values of
(i) $2^{0.5}$; (ii) $2^{0.25}$; (iii) $2^{0.125}$; (iv) $2^{0.0625}$; (v) $2^{0.03125}$
What is the effect of repeating this process indefinitely ?
4. By using tables, write down the values of
(i) $10^{0.5}$; (ii) $10^{0.25}$; (iii) $10^{0.125}$; (iv) $10^{0.0625}$.
What is the effect of repeating this process indefinitely ?
5. If x and nx are small (compared with 1), $(1+x)^n$ is approximately equal to $1+nx$.
Use this fact to find approximate values for
 - (i) $(1 + .01)^{\frac{1}{2}}$;
 - (ii) $\sqrt{1.02}$;
 - (iii) $(1.01)^{-1}$;
 - (iv) $\frac{1}{\sqrt{1.02}}$;
 - (v) $(0.99)^{\frac{1}{2}}$;
 - (vi) $\frac{1}{0.99}$;
 - (vii) $\frac{1}{(1.001)^2}$;
 - (viii) $\frac{1}{\sqrt{104}}$;
 - (ix) $\sqrt[3]{1003}$;
 - (x) $\frac{1}{(1003)^3}$.

6. The “derived function” of x^n with respect to x is nx^{n-1} for all values of n . Use this fact to find the “derived functions” with respect to x of

- (i) x^3 ; (ii) $x^{\frac{3}{2}}$; (iii) x^{-2} ; (iv) $\sqrt{x}$;
 (v) $\frac{1}{x^5}$; (vi) $\frac{1}{\sqrt{x}}$; (vii) x^0 ; (viii) $\frac{1}{\sqrt[3]{x^5}}$.

7. The “integral” of x^n with respect to x is $\frac{1}{n+1} \cdot x^{n+1}$ for all values of n , except $n = -1$. Use this fact to find the integrals with respect to x of

- (i) x^2 ; (ii) $x^{\frac{1}{2}}$; (iii) x^{-2} ; (iv) $\sqrt[3]{x}$; (v) $\frac{1}{x^3}$;
 (vi) $\sqrt[2]{x^3}$; (vii) x ; (viii) x^0 ; (ix) $\frac{1}{\sqrt[3]{x}}$.

8. $\sqrt{3x+1} - \sqrt{x+8} + 1 = 0$. [Test your answers.]

9. (i) $\sqrt{x+1} + \sqrt{x-2} = 3$; (ii) $\sqrt{x+1} - \sqrt{x-2} = 3$.

10. $5 + \sqrt{x-2} = 2x$.

B

1. Using $\sqrt{2} = 1.41$, complete the following table of values for the graph of 2^x from $x = -3$ to $x = +3$:

x	-3	-2	1	0	0.5	1	1.5	2	2.5	3
2^x										

Carefully draw the graph of 2^x . Note that

- (i) 2^x increases without limit as x increases without limit;
 (ii) 2^x is positive for all values of x ;
 (iii) for x positive $2^x > 1$,
 for $x = 0$, $2^x = 1$,
 for x negative $2^x < 1$;
 (iv) if x decreases without limit, 2^x approaches the limit 0.

2. Read from the graph of 2^x approximate values of

- (i) $2^{0.2}$; (ii) $2^{-0.5}$; (iii) $4^{1.1}$.

3. Use the graph of 2^x to solve $2^x = 5$.

4. (i) By using such facts as $10^{0.5} = \sqrt{10}$, $10^{0.25} = \sqrt[4]{10}$, $10^{0.75} = 10^{\frac{3}{4}} = 10^{\frac{1}{2}} \times 10^{\frac{1}{4}}$, etc., compute the values needed for filling in the following table :

$x = 0$	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$10^x =$								

(ii) Hence draw the graph of 10^x from $x = 0$ to $x = 1$.

(iii) From your graph read off the values of $10^{0.2}$, $10^{0.4}$, $10^{0.7}$, $10^{0.9}$.

(iv) From your graph read off the values of x for which
(a) $10^x = 3$; (b) $10^x = 5$; (c) $10^x = 8.1$.

(v) From your graph read off the values of
(a) $\sqrt[5]{10}$; (b) $\sqrt[3]{100}$; (c) $\sqrt[5]{1000}$.

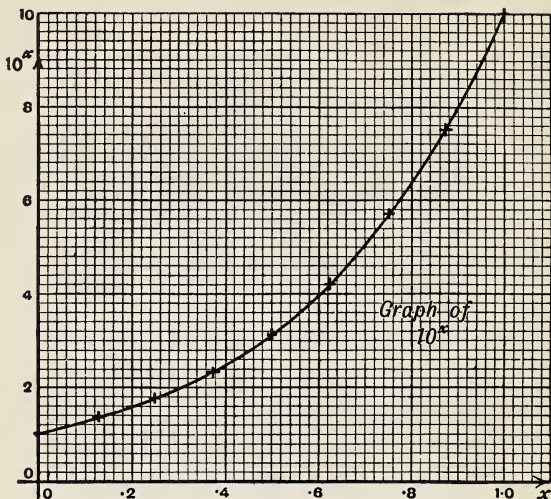
5. Given that $9 = 10^{0.9542}$, express as powers of 10

- (i) 3; (ii) 0.9; (iii) 2.7.

6. The necessary table of values for the graph of 10^x is

$x = 0$	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$10^x = 1$	1.33	1.78	2.37	3.16	4.22	5.62	7.50	10

Plotting them, we have the following graph :



- (a) Extend this graph to negative values of x .
- (b) Compare this graph with the graph of $\log x$.
7. (i) Find from each graph the value of x for which $10^x = 7$.
(ii) What is the logarithm of 7 ?
8. (i) Use each graph to express 3.5 as a power of 10.
(ii) What is the logarithm of 3.5 ?
9. (i) Given that $2 = 10^{0.301}$, express as powers of 10
(a) 20 ; (b) 200 ; (c) 2000 ; (d) 2,000,000.
(ii) What are the logarithms of
(a) 20 ; (b) 200 ; (c) 2000 ; (d) 2,000,000 ?
10. What is the whole number in the index when each of the following numbers is expressed as a power of 10 ?
(i) 73 ; (ii) 7142.3 ; (iii) 8.61 ; (iv) 5.6×10^5 ;
(v) 100,000 ; (vi) 1234 ; (vii) 12,345 ; (viii) 123,456 ;
(ix) 12,345,678 ; (x) 123,456,789.

C

1. Find correct to 3 significant figures the value of

$$\frac{31.7 \times 920000}{415.1 \times 1.6}.$$

$$\text{Rough estimate} = \frac{30 \times 900000}{400 \times 2} = \frac{27 \times 10^6}{800} \approx 3 \times 10^4 \approx 30,000.$$

From the tables, we find the expression

$$\begin{aligned} &= \frac{10^{1.5011} \times 10^{5.9638}}{10^{2.6181} \times 10^{0.2041}} & \begin{array}{r|l} 1.5011 & 2.6181 \\ 5.9638 & 0.2041 \\ \hline 7.4649 & 2.8222 \\ 2.8222 & \\ \hline 4.6427 & \end{array} \\ &= \frac{10^{1.5011+5.9638}}{10^{2.6181+0.2041}} & \\ &= \frac{10^{7.4649}}{10^{2.8222}} = 10^{7.4649-2.8222} & \\ &= 10^{4.6427} & \\ &= 43,920 & \\ &= 43,900 \text{ to 3 sig. fig.} & \end{aligned}$$

Rewrite the above, using logarithmic notation.

Evaluate, correct to 3 significant figures,

2. $\left(\frac{3.18 \times 6.05}{4.45}\right)^2.$

3. $\sqrt[3]{\frac{1.752 \times 753.6}{111.1}}.$

4. (i) $\sqrt[3]{9}$; (ii) $\sqrt[3]{90}$; (iii) $\sqrt[3]{900}$; (iv) $\sqrt[3]{9000}.$

5. $\frac{\sqrt[3]{1.752 \times 753.6}}{111.1}.$

6. $\frac{\sqrt[3]{1.752}}{111.1} \times 753.6.$

7. $\frac{\sqrt[3]{1.752 \times 753.6}}{111.1}.$

8. $\sqrt[3]{1.752} \times \frac{753.6}{111.1}.$

D

1. Simply $(0.0862)^{0.24}$.

$$\begin{aligned}
 (0.0862)^{0.24} &= 10^{\bar{2}.9355 \times 0.24} \text{ or } 10^{(8.9355-10) \times 0.24} && 0.9355 \\
 &= 10^{-0.48+0.2245} && 0.24 \\
 &= 10^{-1+0.52+0.2245} && 18710 \\
 &= 10^{\bar{1}.7445} && 37420 \\
 &= 0.5552. && \underline{\underline{0.22452}}
 \end{aligned}$$

Rewrite the above, using logarithmic notation.

Evaluate to 3 significant figures.

2. $\sqrt[3]{\frac{1}{915.2}}$

3. $\sqrt[3]{\frac{4.156 \times 0.00612}{0.0891}}$

4. $\frac{0.273 + (0.864)^2}{10\frac{1}{4}}$

5. $\frac{5 + \sqrt{8.67}}{10 - \sqrt{86.7}}$

6. Find the value of $\pi(R^2 - r^2)h$ when $R = 7.16$, $r = 3.12$, $h = 2.59$.7. Find the value of $A \cdot e^{kt}$ when $A = 4.7$, $e = 2.718$, $k = 2.4$, $t = \frac{1}{2}$.

8. What sum invested for a child at birth will amount to \$1000 when he comes of age if it is invested at 6 per cent. compound interest?

9. Find x , given that $4.7^x = 0.83$.

$$\log (4.7)^x = \log 0.83 ;$$

$$\therefore x \log 4.7 = \log 0.83 ;$$

$$\therefore x = \frac{\log 0.83}{\log 4.7} = \frac{\bar{1}.9191}{0.6721} \text{ or } \frac{9.9191 - 10}{0.6721}$$

$$= \frac{-1 + .9191}{0.6721} = -\frac{0.0809}{0.6721}$$

$$= -\frac{10^{\bar{2}.9079}}{10^{\bar{1}.8275}}$$

$$= -10^{\bar{1}.0804}$$

$$= -0.1203 ;$$

$$\therefore x = -0.120.$$

$$\begin{aligned}
 &2.9079 \\
 &\bar{1}.8275 \\
 &\underline{\underline{\bar{1}.0804}}
 \end{aligned}$$

Rewrite the above, replacing $\bar{1}$ by 9-10 and $\bar{2}$ by 8-10 wherever each occurs. Verify result.[Unless otherwise stated, $\log x$ means $\log_{10} x$.]

10. Simplify

(i) $\log x + \log \frac{1}{x}$; (ii) $\log \frac{x}{y} + \log y$.

11. Express the following in a form which does not involve the logarithmic notation :

(i) $\log x + \log y = \log 5$; (ii) $\log x + 3 \log y = 2$;
 (iii) $x \log 3 = 3$; (iv) $3 \log x - 2 \log y = 1$;
 (v) $x \log 5 = y \log 6$; (vi) $x \log 5 = y \log 6 + 2$.

12. Solve the equations

(i) $2^x = 8192$; (ii) $14 \cdot 5^n = 210 \cdot 3$; (iii) $7 \cdot 81^n = 0 \cdot 128$.

13. If $pv^n = 475$, find n when $p = 3 \cdot 62$, $v = 98 \cdot 5$.

E

1. If an oak beam l feet long, b inches wide and d inches thick is fixed at one end, and a weight W cwt. is placed at the other end, it will break if W is larger than $\frac{5bd^2}{4l}$. Find whether such a beam 10 feet long, 1 foot wide and 9 inches thick will break when a load of 6 tons is placed at the free end.

(i) $(3x)^2 - 3x^2$. (ii) $\left(a + \frac{b}{2}\right) - \left(\frac{a+b}{2}\right)$.

2. Make w the subject of the formula $P = \frac{W + nw}{8}$.

3. Which of the numbers 0, 1, 2, 3, 4 (if any) are roots of the equation $(2 - x)^5 + (4 - x)^5 = (6 - 2x)^5$?

4. A farmer has 60 hurdles, each 2 yards long, with which to make a rectangular enclosure, of which one side is already formed by the fence of the field. Complete the following table :

Number of hurdles in side parallel to fence - -	10	20	30	40	50	60
Number of hurdles in each of the other sides - -	25	20	15	10	5	0
Area of enclosure in square yards - - - -	1000	1600	1800	1600	1000	0

Draw a graph showing how the area depends on the number of hurdles placed in the side parallel to the fence, and find the number which gives the largest area possible.

5. Find the value of a , if $x = 2$ is a solution of the equation

$$3x^2 + 2ax - 20 = 0.$$

6. Draw a graph to illustrate the relation between P and W , if $P = \frac{1}{5}W + 4.6$, for values of W from 0 to 40. Read off the value of W when P is 10.

7. Prove that a triangle whose sides are $(m^2 + n^2)$ inches, $(m^2 - n^2)$ inches, and $2mn$ inches is a right-angled triangle with $(m^2 + n^2)$ as its longest side whatever numbers m and n are. Give a numerical result when $m = 5$ and $n = 3$.

8. An effort of P lb. is required to lift a weight of W lb. by a certain machine, where P and W are known to be connected by the equation $P = aW + b$, where a and b are constants, *i.e.* independent of the values of P and W . It is found that when $W = 10$, $P = 15$, and when $W = 14$, $P = 16.2$. Find the values of a and b . What is the value of W when $P = 9$, and what does this mean?

9. The weight of a foot of iron piping is $2.45(D^2 - d^2)$ lb., where D is the external and d is the internal diameter of the pipe in inches.

(i) Find the internal diameter of a pipe of external diameter 5.2 inches, which weighs 9.8 lb. per foot length; (ii) Express the weight in terms of the mean radius r inch and the thickness t inch.

10. A man walks a certain distance, x miles, up hill at 3 miles an hour, and walks back at 6 miles an hour. How long does he take going uphill? How long does he take coming back? How far has he been altogether, and how long has he taken? Show that his average speed for the whole journey is 4 miles an hour.

[Lewis Carroll.]

11. A rectangle l inches long and b inches broad is such that its area would be increased by 44 sq. inches, if its length and breadth were both increased by 1 inch. Express this fact by an equation connecting l and b , and prove that the perimeter of the rectangle is 86 inches.

12. Simplify the following expressions :

$$(i) \frac{1}{h} \left[\frac{1}{x+h} - \frac{1}{x} \right], \quad (ii) \frac{1}{h} \left[\frac{1}{(x+h)^2} - \frac{1}{x^2} \right],$$

and find approximately their values when $x = 1$, $h = 0.001$.

13. Find the value of the constants a and b in the formula

$$\frac{1}{w} = \frac{a}{u} + \frac{b}{v}$$

if $w = 5$, when $u = 4$ and $v = 10$; and $w = 6$, when $u = 3$ and $v = 9$.

F

1. The following observed values of x and y will give a straight-line graph, subject to the errors of experiment :

x	1	2	3	4	5	6	7	8
y	7.2	12.1	16.7	21.8	26.6	31.5	36.3	41.2

Plot these values and draw the straight line which fits them best. Read off the most probable value of y when $x = 4.7$, and of x when $y = 30$.

2. Write the following numbers without using the index notation :

$$(i) 1.8 \times 10^6, \quad (ii) 1.8 \times 10^{-2}, \\ (iii) 1.8 \times 10^{-4}, \quad (iv) 1.8 \times 100^{\frac{3}{2}}.$$

and express 73800 and 0.0000738 in this form.

3. The equation $s = a + bt + t^2$ is satisfied by the two pairs of values $t = 3$, $s = 5$ and $t = 5$, $s = 29$.

Find the values of a and b , if they are constants.

4. Find the value of $P^{\frac{1}{2}}H^{-\frac{1}{2}}$, when $P = 144$, $H = 16$.

5. The efficiency, e , of a certain machine is given by the formula $e = \frac{W(a-b)}{2P \cdot a}$. Express a in terms of e , P , W and b .

6. If $a = K \cdot H^{1.25} \cdot P^{-0.5}$ express P in terms of a , K and H .

7. Find the value when $a = 2$, $b = -1$, and $n = 3$ of the following expressions :

$$\begin{array}{lll} \text{(i)} (a + b)^n. & \text{(ii)} a^n + b^n. & \text{(iii)} (a + b)^{n-1}. \\ \text{(iv)} a^{n-1} + b^{n-1}. & \text{(v)} \frac{a^n}{b^n} \div \frac{a^{n-1}}{b^{n-1}}. \end{array}$$

What will be the value of (v) when $n = 10$?

8. Find the value of $\sqrt{\frac{H \cdot D^5}{L}}$, correct to 3 figures, if $H = 25.7$, $D = 2.32$, $L = 880$.

9. Find the value when $n = 3$ of

$$\text{(i)} 8^{\frac{1}{n}}; \text{(ii)} 8^{-n}; \text{(iii)} 8^{1-\frac{1}{n}}; \text{(iv)} 16^{1-n}; \text{(v)} 16^{n-3}.$$

10. Two quantities s and t are connected by the formula

$$s = 6t - 4t^2.$$

Tabulate the values of s , when t has the values $-2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2$. Draw a graph to show the relation between s and t , and from it read off the largest value of s , and also the value of t when $s = 1$.

$$11. \text{ Prove that } \frac{n^2(n+1)^2}{4} + (n+1)^3 \equiv \frac{(n+1)^2(n+2)^2}{4}.$$

[Note that $(n+1)^2$ is a factor of the left-hand side of this identity.]

Write down (without multiplying out) the special cases of this identity for $n = 0, 1, 2, 3, 4, 5, 6$. By adding these results together, prove that $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 = \frac{7^2 \times 8^2}{4}$.

By generalising this method, find an expression for the sum of the cubes of the first n integers.

12. Solve the equations :

$$\begin{array}{l} \text{(i)} 3x^2 - 2x - 8 = 0; \\ \text{(ii)} 3x^2 - 2x - 7 = 0, \end{array}$$

giving the result of the latter correct to 3 figures.

13. Find the value of pv^n when $p = 14.21$, $v = 7.7$ and $n=4$, correct to 3 figures.

$$\begin{aligned}
 14. \quad & 3 \times 4 = 12 ; & \therefore 35^2 = 1225. \\
 & 4 \times 5 = 20 ; & \therefore 45^2 = 2025. \\
 & 7 \times 8 = 56 ; & \therefore 75^2 = 5625. \\
 & 12 \times 13 = 156 ; & \therefore 125^2 = 15625.
 \end{aligned}$$

What general formula covers these facts ? Prove it.

15. The bore, d inches, of a pipe through which a pump of horse-power H can deliver G gallons per second is given by

$$d = 1.25 \cdot G^{\frac{1}{3}} H^{-\frac{1}{4}}.$$

- (i) Find the bore necessary for an engine of 15 H.P. to deliver 300 gallons per minute.
 (ii) Rewrite the formula, making G the subject. (C.S.C.)

16. In Fig. 53, AB represents a line 10 inches long, and P is a point such that $AP = x$ inches. Squares are described on AB , AP and PB as shown.

Write down the areas of these squares and express the area of the shaded portion of the figure in terms of x .

Tabulate the values of this area as x varies from 0 to 10, and show the results on a graph. Find from your graph the position of P which makes the shaded area greatest.

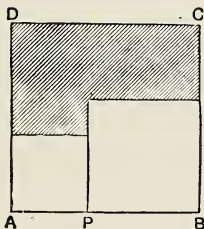


FIG. 53.

17. Find, without using tables, the logarithms of

$$\sqrt[4]{10}, \quad \sqrt[4]{0.1}, \quad \sqrt[5]{0.001}, \quad \sqrt[3]{0.1}.$$

18. $(1 + x)^n \simeq 1 + nx$, when x and nx are small. Use logarithms to find the error per cent. in assuming that

$$(1 + x)^n = 1 + nx,$$

when $x = 0.025$ and $n = 4$.

19. A small object 3 mm. long when seen through a magnifying glass held d mm. from it, appears to have a length l mm. given by $l = \frac{3f}{f-d}$, where f is a constant depending on the kind of magnifying glass used.

(i) Find the value of f for a certain magnifying glass in which the apparent length l is 12 mm., when the glass is held 15 mm. away.

(ii) Find also the apparent length of the same object seen through the same glass, when the glass is held 12 mm. away.

20. The safe width of a dam at a depth of x feet below water level is $\sqrt{\left\{ \frac{0.05 \cdot x^3}{9 + 0.03x} \right\}}$ feet.

How wide should a dam be at a depth of 28 feet below water level ? (Certificate.)

21. Evaluate the following as shortly as possible, given that $\sqrt{2} = 1.414 \dots$, $\sqrt{3} = 1.732 \dots$:

$$(i) \frac{4}{\sqrt{2}}; \quad (ii) \frac{6}{\sqrt{2}}; \quad (iii) \frac{\sqrt{6}}{\sqrt{3}}; \quad (iv) \sqrt{12}.$$

22. In a sailing-boat the direction of the wind makes an angle α with the direction of the boat's keel, and the sail makes an angle β with the boat's keel. The following table shows the best angle to choose for β when sailing into a wind :

α in degrees	-	60°	50°	40°	30°	20°	10°
β in degrees	-	$34\frac{1}{2}^\circ$	28°	22°	16°	10°	5°

Exhibit these results on a graph, and read off from the graph the best angle at which to set the sail when the course required is due North and the wind is blowing from the North-East.

23. Find x if (i) $3^x = 81$; (ii) $81^x = 3$; (iii) $3^x = \frac{1}{9}$; (iv) $81^{-x} = 3$.

24. Find the value of (i) 6^{20} ; (ii) 2^{60} , correct to one significant figure, in the approximate form $a \times 10^b$, a being an integer less than 10.

25. Write 27 , $\frac{1}{9}$, $3\sqrt{3}$ and $\sqrt[4]{3}$ as powers of 3 , and write down their logarithms to the base 3 .

26. A certain sheet of transparent material absorbs $\frac{1}{10}$ th of the light of a certain colour falling on it. What fraction of the light of that colour gets through two such sheets?

How many sheets must be placed one over another to reduce the light of that colour by 60 per cent.?

G

1. Find the values of $(1-x)(x+2)$ when x has integral values from -3 to $+3$. Plot a graph to illustrate these figures, and read off from it the greatest value of $(1-x)(x+2)$ for all values of x in this interval.

2. Write the formula $W = 17.6 \frac{PR^2}{r}$ with R as the subject, and find the value of R , correct to 3 figures, when $W = 4480$, $r = 0.29$, $P = 9.6$.

3. The "displacements" of similar ships vary as the cubes of their lengths. Find the displacement of a ship which is half as long again as a similar ship whose displacement is 2000 tons.

4. The time of oscillation of a simple pendulum varies as the square root of its length. If the time for a pendulum 2 feet long is $1\frac{1}{2}$ seconds, find the time of oscillation of a pendulum 1 yard long (correct to two figures).

5. Show that a triangle whose sides are $\sqrt{5} - 1$, $\sqrt{5} + 1$ and $2\sqrt{3}$ inches is right-angled, and find its area.

6. It is found that the current required to fuse a wire of given material varies as $D^{\frac{3}{2}}$, where D is the diameter of the wire. A current of 154.1 amperes fuses a leaden wire of diameter 0.232 inch; find in amperes, correct to 3 figures, the current which will fuse a leaden wire of diameter 0.316 inch.

(Certificate.)

7. Simplify (i) $\frac{2^{n+1} + 2^n}{2^n - 2^{n-1}}$; (ii) $\left(\frac{125}{8}\right)^{-\frac{2}{3}}$.

8. The following table gives values for the pressure and volume of a mass of gas kept at constant temperature :

Volume in c.c. -	47.1	44.0	42.4	40.4	38.4	36.3
Pressure in atmospheres - -	1.47	1.57	1.63	1.71	1.80	1.90

Illustrate these figures by a graph, and read off the volume when the pressure is 1.85 atmospheres.

9. If d is the mean distance of a planet from the sun, then the time of its revolution round the sun or its "year" varies as $d^{\frac{3}{2}}$. If the mean distances of Jupiter and the Earth from the sun are roughly in the ratio 16 to 3, find the length of Jupiter's "year" in days.

10. If y is known to vary inversely as x , complete the following table :

x	4	2	1	$\frac{1}{2}$			0.1	
y	3	6			15	20		100

11. The resistance of a wire varies as the length of the wire and inversely as the square of the diameter. Compare the resistances of two copper wires, one 100 feet long and of diameter $\frac{1}{16}$ inch, the other 50 feet long and diameter 0.1 inch.

12. Given $P = aW + b$. If $P = 14$ when $W = 6$, and $P = 17.2$ when $W = 8$, find the values of a and b .

H

1. (i) For what values of x is the function $(x - 5)(x - 6)$ equal to 0 ?
- (ii) What sort of value does this function have (a) when x is large and positive, say $+100$? (b) when x is large and negative, say -100 ?
- (iii) Between what values of x is this function negative, and what can you say about its value then ?
- (iv) Sketch roughly without squared paper the graph of this function.

2. (i) Find a value of x for which the function $\frac{1}{x-6}$ is

- (a) positive and < 0.001 ,
- (b) negative but > -0.001 .

(ii) Find a value of x for which this function is

- (a) positive and > 1000 ,
- (b) negative and < -1000 .

3. Make a table of values of the function $3 + 4x - x^2$ for values of x from -1 to 5 . Draw an accurate graph of the function between these values of x , and from it read off the answers to the following questions :

- (i) What is the maximum value of the function ?
- (ii) For what value of x is the function a maximum ?
- (iii) Between what values of x is the function positive ?
- (iv) For what values of x is the function equal to 4 ?
- (v) Solve the equation $3 + 4x - x^2 = 6$.
- (vi) Solve the equation $x^2 - 4x - 4 = 0$.

4. The force of attraction between two magnetic poles varies inversely as the square of the distance between them. If two magnetic poles attract each other with a force of 5.6 dynes, when 5 cm. apart, find the force of attraction between them when they are 7.5 cm. apart.

5. For what values of x is the function $\frac{x^2}{100} = 0$? Can the function $\frac{x^2}{100}$ ever be negative ? What is its value when x is large, *e.g.* when $x = \pm 100$? Sketch roughly the graph of this function.

6. The volume of a gas varies as the absolute temperature and inversely as the pressure. When the pressure is 15 lb. per sq. inch and the absolute temperature is 280° , the volume of a gas is 200 cub. inches. Find its volume when the pressure is 10 lb. per sq. inch and the absolute temperature 300° .

7. Plot the graph of the function $\frac{x^2}{2} + \frac{1}{2x}$ from $x = 3$ to $x = -3$, and from it read off the roots of the equation $\frac{x^2}{2} + \frac{1}{2x} = 3$.

8. What is the value of the function $\frac{x}{x+1}$, (a) when $x \rightarrow +\infty$, (b) when $x \rightarrow -\infty$, (c) when $x \rightarrow -1$, but is < -1 , (d) when $x \rightarrow -1$, but is > -1 ? For what value of x is the function zero? Draw a rough graph of this function without using squared paper.

9. Find the value of $4.75H^3d^{-4}$, when $H=1.723$, $d=0.1614$.

10. Sketch roughly the graph of the function $x(x-2)(x-4)$.

11. Draw accurate graphs of the functions $\frac{x^2}{10}$ and $\frac{1}{x}$ on the same piece of paper and with the same scales, for values of x between 6 and -6 .

For what values of x are these two functions equal? Read off this value, (i) from the graph, (ii) using log tables.

12. If $f(x) = x^2 + 7x - 6$, find the values of $f(1)$, $f(0)$, $f(2a)$ and $f(x+1)$.

13. Simplify $\frac{3(x+h)^3 - 3x^3}{h}$, and find its value when $x=1$, $h=0.001$, correct to 3 figures.

14. If $f(x) = x^2 - \frac{10}{x}$, what is the value of $f(1)$, $f(10)$, $f(0.1)$? Has $f(0)$ any meaning? What are $f(-0.1)$, $f(-0.01)$?

15. (i) When is the function $\frac{x-3}{x+3}$ equal to zero?

(ii) Find values of x for which this function is (a) > 1000 ,
(b) < -1000 .

(iii) To what value does this function tend

(a) when $x \rightarrow +\infty$?

(b) when $x \rightarrow -\infty$?

(iv) Sketch the graph of this function.

16. The following table gives the heights in feet of certain points on a road above sea-level, and their horizontal distances in yards from the starting point:

Height h	100	112	127	139	140	138	130
Distance d	0	50	100	150	200	250	300

Show these results on a graph. Calculate the average gradient

- (a) for the first 50 yards horizontally,
- (b) for the first 100 yards horizontally.

By drawing a tangent, estimate the gradient at the point 100 yards distant from the starting point.

17. An underground train travels from one station to the next, a distance of 1000 yards, in 80 seconds, in accordance with the following table :

Time in seconds -	10	20	30	40	50	60	70	80
Distance travelled in yards - - -	50	135	260	435	610	780	900	1000

Draw a graph to illustrate these numbers.

- Calculate (i) the average speed for the whole journey ;
- (ii) the average speed for the first 20 seconds ;
- (iii) the average speed for the period 20 sec. to 40 sec.

By drawing a tangent, estimate the actual speed 20 seconds after the train started. Give all results in the form feet per second.

18. If $x^3 = 175t^2$ and $t^5 = y$, express x in terms of y only, and find the value of x , when $y = 0.0162$, correct to 3 figures.

19. Plot values of the function $6x - x^2$ as x varies from -1 to $+7$, and draw an accurate graph.

What is the change in the function per unit change in x

- (i) as x changes from 1 to 3 ?
- (ii) as x changes from 1 to 1.1 ?
- (iii) as x changes from 1 to $1 + h$?

What is the limit of the last expression as $h \rightarrow 0$?

Draw a tangent to the graph at the point given by $x = 1$; estimate its gradient, and compare with the result obtained by calculation.

20. If $f(x) = x^2 + 6$, find the value of $f(1)$ and $f(-1)$.

Write down the value of $\frac{f(x+h) - f(x)}{h}$, and simplify this expression. To what limit does it tend, when $h \rightarrow 0$?

21. Make a table of values of the function $y = x^2 + 2x - 3$ from $x = -4$ to $x = +2$; draw an accurate graph, and read off answers to the following questions:

- (i) What is the minimum value of the function?
- (ii) For what value of x is the function a minimum?
- (iii) Between what values of x is the function negative?
- (iv) For what values of x does the function equal 4?
- (v) Solve the equations: (a) $x^2 + 2x - 3 = 1$,
(b) $x^2 + 2x = 2$.

22. With the function and graph of Question 1, what is the average change in x per unit change in y ,

- (i) when x changes from -4 to -2 ?
- (ii) when x changes from -3 to -2 ?
- (iii) when x changes from $-(2+h)$ to -2 ?
- (iv) What is the limit of this last expression as $h \rightarrow 0$?
- (v) Draw a tangent to the graph at the point given by $x = -2$, and estimate its gradient. Compare with the answer to part (iv).

23. If $y = f(x) = \frac{ax+b}{cx-a}$, prove that $x = f(y)$.

24. If $f(x)$ denotes the greatest integer in x , sketch the graph of (i) $f(x)$, (ii) $x - f(x)$, when x is positive.

25. Make a table of values of the function $\frac{1}{10}(x^3 - 4x)$ for values of x from -4 to $+4$; draw an accurate graph of the function, and read off answers to the following questions:

- (i) For what positive value of x is the function a minimum?
- (ii) What is the maximum value of the function for negative values of x ?
- (iii) For what values of x is the function negative?
- (iv) For what values of x does the function equal 0.2 ?
- (v) Solve the equations (a) $\frac{1}{10}(x^3 - 4x) = 0.1$,
(b) $x^3 - 4x = 1$,
(c) $x^3 - 4x + 1 = 0$.

26. Find the gradient of the graph of Ex. 1 at the point given by $x = 2$,

- (a) by drawing a tangent to the graph,
- (b) by calculation.

27. Sketch roughly the graph of the function $(x-4)^2(x-3)$.

28. $f(x)$ is a function of x such that $f(x) \times f(y) = f(x + y)$, for all values of x and y ; and $f(1) = 2$. Prove that

$$f(2) = 4, f(3) = 8,$$

and find the values of $f(0)$ and $f(\frac{1}{2})$.

What is the simplest form of this function of x ?

I

1. Soundings are made on a sea-shore at low tide to discover the shape of the sea-bottom, and the following measurements are recorded :

Distance from low water mark in yards - - - -	50	100	150	200	250	300
Depth of water in fathoms -	0.7	2	4	4.5	5	5.1

Draw a graph to illustrate these figures.

Estimate the average gradient of the sea-bottom

- for the first 300 yards out,
- between points 100 yards and 250 yards out.
- Draw a tangent and find the gradient at a point 150 yards out. [1 fathom = 6 feet.]

2. Find (i) $\text{Lt}_{h \rightarrow 0} \frac{1}{h} \{(x + h)^2 + 4(x + h) - x^2 - 4x\}$.

(ii) $\text{Lt}_{h \rightarrow 0} \frac{1}{h} \left\{ \frac{1}{x + h - 1} - \frac{1}{x - 1} \right\}$.

3. Find, by any method, the gradient of the graph of the function $x^3 - 5x$

- at the point where $x = 1$,
- at the point where $x = a$.

4. Sketch roughly a graph of the function $(x - 2)^3(x - 4)$.

5. Draw roughly with the same axes, the graphs of

- $y = 3 - x$.
- $y = 5 - x$.
- $y = 5$.

What are the gradients of these lines?

6. Differentiate the following functions with respect to x .

- | | |
|-------------------------|--|
| (i) $4x - 3x^2$. | (ii) $3 + 4x - 3x^2$. |
| (iii) $6 + 4x - 3x^2$. | (iv) $\frac{3}{2}x + \frac{2}{5}x^2$. |
| (v) $6x^{-1}$. | (vi) $8\sqrt{x}$. |

7. (i) Find the gradient of the graph $y = 7 + 6x + 3x^2$ at the point $x = 2$.

(ii) At what point is the gradient equal to 1 ?

8. y is known to be a function of x of the type $mx + c$, m and c being constants. If $y = 43$ when $x = 17$, and $y = 34$ when $x = 14$, express y as a function of x .

9. Find $\frac{dy}{dx}$ (i) when $y = 4x^2 - 3 + \frac{1}{x}$,
 (ii) when $y = 16x^3 - 4x^2 + 2x - 5$,
 (iii) when $y = 5x^{\frac{1}{2}} + x^{-\frac{1}{2}}$.

10. Find the gradient at $x = a$ of the graph

$$y = x^3 - 6x^2 - 15x + 5.$$

Find for what values of x the gradient of this graph is zero. Sketch the graph roughly.

11. The distance, s feet, of a particle moving in a straight line from a fixed origin is given by $s = 4 + 6t + 3t^2$, where t is the number of seconds for which it has been moving. Find its velocity, (i) 2 seconds, (ii) 10 seconds after it started.

12. Find (i) $\text{Lt}_{h \rightarrow 0} \left\{ \frac{\frac{1}{(x+h)^2 - 2} - \frac{1}{x^2 - 2}}{h} \right\}$
 (ii) $\text{Lt}_{n \rightarrow \infty} \frac{n^2 + 1}{2n^2 + 1}$.

13. Differentiate the following with respect to x :

- | | | | |
|-------------------|--------------------|-------------------------|------------------------------|
| (i) $\sqrt{6x}$; | (ii) $6\sqrt{x}$; | (iii) $\frac{4}{x^3}$; | (iv) $x^2 + \frac{1}{x^2}$. |
|-------------------|--------------------|-------------------------|------------------------------|

14. For what values of x is the function $x^3 - 12x$ equal to zero ? Find its derived function and find for what values of x the derived function is zero. From these data sketch a rough graph of the function.

15. Differentiate the following functions of t with respect to t :

$$(i) 4t + \frac{4}{t}; \quad (ii) 3t^3 - 4t^{-1} + 5t^{-3}; \quad (iii) 5t^{\frac{5}{2}}.$$

16. Differentiate the function $x^3 - 27x$. For what values of x is its rate of change zero? What are the maximum and minimum values of this function. Sketch its graph.

17. The weight of a certain solid is given by the formula

$$10x^2(15 - 2x),$$

where x is variable. Show that as x increases from 0 to 5 the solid gets steadily heavier. What happens after that?

18. If $f(x) = x^2 + 3x$, find the value of

$$(i) \frac{f(x+h) - f(x)}{h}.$$

$$(ii) \frac{f(x+h) - f(x) - \{f(x) - f(x-h)\}}{h^2}.$$

Find the limits to which these two expressions tend, as $h \rightarrow 0$.

19. Find the turning points of the function $x^3 + 3x^2 - 9x + 6$ and determine whether they are maxima or minima.

20. Show that the function $x^3 + 5x - 6$ steadily increases as x increases.

One root of the equation $x^3 + 5x - 6 = 0$ is $x = 1$. Show why it cannot have another real root.

21. A piece of wire 8 inches long is bent so as to form a rectangle. If one side of the rectangle is x inches, express the area as a function of x , and find the value of x for which the area is a maximum.

22. Differentiate the function $5 - 3x - 2x^3$ and show that the function steadily decreases as x increases, and that it can be zero only when $x = 1$. Draw roughly the graph of this function.

23. Tabulate the values of the function $x + \frac{4}{x}$, as x varies from -4 to $+4$. Draw an accurate graph of this function and read off the values of x for which this function is a maximum or minimum.

Verify your answers by differentiating the function, and calculating the values of x required.

24. Draw a tangent to the graph (Question 26) at the point $x = 3$, and estimate its gradient. Check your answer by calculating the value of $\frac{dy}{dx}$ when $x = 3$.

25. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$,

- (i) when $y = 3x^3 - 7x$; (ii) when $y = 5x^2 - 4$;
 (iii) when $y = 7x + 6$; (iv) when $y = 5$;
 (v) when $y = \frac{6}{x}$.

26. Draw a rough sketch of the graph of the function $75x - 4x^3$ and find the maximum and minimum ordinates.

27. (i) Is $\frac{x^2 - 1}{x - 1}$ always equal to $x + 1$?

(ii) Can you find a value of x for which $\frac{x^2 - 1}{x - 1}$ equals 2?

(iii) Find $\text{Lt}_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$.

28. A bullet fired vertically upwards rises s feet in t seconds, where $s = 500t - 16t^2$. (i) Find its height after 10 sec., 11 sec., 10.1 sec.; (ii) What is its average speed in the intervals 10 to 11 sec. and 10 to 10.1 sec.? (iii) What is its average speed in the interval 10 to $10 + h$ sec.? (iv) What is its velocity after 10 sec.?

29. If y varies as x^3 , complete the table:

$x = 2$	4	30
$y = 5$		

30. The sides of a triangle are 5, 5, $2x$ cm.; its area is A sq. cm. Express A as a function of x . Represent this function by a graph and find from the graph the maximum area of the triangle.
 (C.S.C.)

31. Ox , Oy are horizontal and vertical lines : the graph of $y = x^2$ represents a hill side on the scale, unit for x -axis = 100 yards, unit for y -axis = 1 foot. What is the average slope of the hill, (i) from $x = 1$ to $x = 2$; (ii) from $x = 1$ to $x = 1.1$; (iii) from $x = 1$ to $x = 1 + h$. What is the gradient of the hill at $x=1$?

32. An open cardboard box with square ends is fitted with an overlapping lid which covers the open top, the whole of two square ends and one other side. The total area of the cardboard is 8 sq. feet. What is the maximum volume of the box ?

33. The highest and lowest marks in an examination were 72 and 17. These were scaled so that the highest and lowest were 200 and 100. What was the mark corresponding to an original mark of 53 ?

34. The electromotive force of a certain type of cell has been found to vary with the temperature as follows :

Temperature (Centigrade)	15	20	25	30
E.M.F. in volts - -	1.4340	1.4284	1.4233	1.4188

What is the average change in E.M.F. per degree of temperature, when the temperature rises from (i) 15° C. to 30° C. ; (ii) 15° C. to 25° C. ; (iii) 15° C. to 20° C. ? Illustrate the table by a graph and by drawing a tangent estimate the rate of fall of E.M.F. at 20° C.

35. Find $\frac{dy}{dx}$ if (i) $y = \frac{x^3 - 1}{x}$; (ii) $y^3 = 2x^2$.

36. (i) Plot the function $x + (-1)^x$ for positive integral values of x .

(ii) What is the least integral value of x , for which and for all greater values of x , $x + (-1)^x > 100$?

(iii) What is $\text{Lt}_{x \rightarrow \infty} [x + (-1)^x]$?

37. The following table gives the best wheel bases for trucks which have to travel on a curved track :

Radius of curve in feet= r	20	30	40	50	60
Wheel base in inches= b	66	69		75	78

What simple relation connects b and r ? Complete the table.

38. (i) Find the circumference of a circle whose area is 3.85 sq. inches.

(ii) The area of a circle is x sq. inches, the circumference is y inches ; express y as a function of x .

39. For what values of x is the function $108x - x^4$, (i) positive, (ii) an increasing function, (iii) a maximum ? Sketch its graph.

40. Kepler's Third Law states that the square of the time a planet takes for one revolution round the sun varies as the cube of its mean distance from the sun. In the following table, the mean distance of the earth from the sun (*i.e.* 93,000,000 miles) is taken as the unit of distance.

	Earth	Mercury	Venus	Mars	Jupiter	Uranus
Time in years	1	0.24	0.62	1.88		
Mean distance	1	0.38		1.52	5.20	19.18

Verify Kepler's Law and complete the table.

41. Use logarithm tables to evaluate $x \times \delta(\log x) \div \delta x$, when (i) $x = 5$, $\delta x = 0.1$; (ii) $x = 7$, $\delta x = 0.1$; (iii) $x = 9$, $\delta x = 0.1$. What do you infer from these results ?

42. Draw freehand the graph of a function $y = f(x)$ for which

(i) over a portion AB $\frac{dy}{dx}$ is positive and $\frac{d^2y}{dx^2}$ is negative ;

(ii) over a portion CD $\frac{dy}{dx}$ is negative and $\frac{d^2y}{dx^2}$ is positive.

43. If V varies directly as the square of x and inversely as y , complete the given table.

		Values of x .		
		1	2	3
Values of y	1			
	2			270
	3		80	

44. From the given table evaluate $\frac{1}{10^x} \times \delta(10^x) \div \delta x$ for $10^x = 5$ and $10^x = 6$ and $10^x = 8$:

10^x	5	5.0001	6	6.0001	8	8.0001
x	0.6989700	0.6989787	0.7781513	0.7781585	0.9030900	0.9030954

What do you infer from these results ?

45. If $xy = 5$, find

(i) $\frac{dy}{dx}$ in terms of x ; (ii) $\frac{dx}{dy}$ in terms of x ; (iii) $\frac{dy}{dx} \times \frac{dx}{dy}$.

46. If $f(x) = x^2 - 3x + 5$, (i) express $\frac{f(x+h) - f(x)}{h}$ in terms of x, h ; (ii) find the limit of this expression when $h \rightarrow 0$; (iii) for what value of x is this limit zero ; (iv) interpret the results of (i), (ii), (iii) geometrically.

47. The function $y = 3x + \frac{a}{x}$ decreases as x increases from 0 to 5, and increases as x increases from 5 to 10. (i) What is the value of a ? (ii) what is the value of $\frac{d^2y}{dx^2}$ when $x = 5$ and $x = -5$? (iii) find maximum and minimum values of the function, distinguishing between them ; (iv) explain your results by a sketch of the function.

J

1. (i) Describe the changes of sign in the function

$$\frac{3(x-2)(x-4)}{(x-1)(x-5)}$$

when x increases from -1 to 6 .

- (ii) Find correct to one significant figure its value when $x = 1.0001$.
- (iii) Can you find a value of x for which the function equals 3 ?
- (iv) Find by logarithms the value of the function when $x = 1000$.
- (v) Simplify $\frac{3(x-2)(x-4)}{(x-1)(x-5)} - 3$, and evaluate $\lim_{x \rightarrow \infty} \frac{3(x-2)(x-4)}{(x-1)(x-5)}$.
- (vi) Sketch the graph of the function for positive values of x .

2. (i) Solve $\frac{dy}{dx} = x - \sqrt{x}$, given $y = 1$ when $x = 1$.

- (ii) Solve $\frac{d^2y}{dx^2} = \frac{1}{x^3}$, given, $y = 4$ when $x = \frac{1}{2}$, and $y = -2$ when $x = -\frac{1}{2}$.

3. Find the values of

$$(i) \int \left(5x^2 + \frac{4}{x^2} \right) dx; \quad (ii) \int \left(x^{1.2} + \frac{1}{x^{1.2}} \right) dx.$$

4. Interpret geometrically $\int_1^4 (2x + 3) dx$; evaluate it (i) by direct calculation and (ii) geometrically.

5. The volume of a bowl of depth x cut from a sphere of radius a is $\pi \int_{a-x}^a (a^2 - x^2) dx$. Compare the volumes of two bowls of depths $\frac{a}{2}$ and $\frac{a}{3}$.

6. (i) One solution of the equations $x^2 + y^2 = 25$, $\frac{x^2}{9} + \frac{y^2}{16} = 2$ is $x = 3$, $y = 4$; write down the other three solutions.
- (ii) If $x^2 + y^2 = 25$, what are the greatest and least values of x , if y is real?
- (iii) If $\frac{x^2}{9} + \frac{y^2}{16} = 2$, what are the greatest and least values of y if x is real?
- (iv) Sketch the graphs of $x^2 + y^2 = 25$ and of $\frac{x^2}{9} + \frac{y^2}{16} = 2$.

7. (i) Simplify the inequality $\frac{x-1}{2} > 7$.
- (ii) Within what limits must x lie if $x^2 < 3x$?

8. (i) Sketch the graph of $y = (1-x)(x-5)$. Calculate the gradient of the tangents at the points where it crosses the x -axis, and find where the gradient is zero.
- (ii) Find the area of the portion of the curve which lies on the positive side of the x -axis.

9. (i) Draw the graph of $x + 2y = 3$.
- (ii) How can you represent graphically all pairs of values of x, y such that $x + 2y > 3$?
- (iii) Represent graphically the pairs of values of x and y for which simultaneously $x + 2y > 3$, $x - y < 1$, $5y - 2x < 10$.
- (iv) Hence obtain all the integral values of x and y which satisfy the inequalities in (iii).

10. Given that $2 = 10^{0.3010}$ and $3 = 10^{0.4771}$, express as powers of 10 without using tables, (i) 1.2; (ii) 250; (iii) 0.015; (iv) 6.4.

11. (i) Sketch the graph of $y = x(x-1)(x-2)$ for values of x from -1 to $+3$.
- (ii) Prove that $\int_0^2 x(x-1)(x-2)dx = 0$, and interpret this result geometrically.
- (iii) Find the minimum and maximum values of y .

12. If a force of P lb. acting on a body moves it s feet in the direction of the force, then the "work done" by the force is $\int_0^s P \cdot ds$ ft.-lb. Find the work done by a variable force $P = 3s + 2$, which moves a body 10 feet.

13. The volume of a gas varies directly as its absolute temperature and inversely as its pressure. At an absolute temperature of 300° , the volume is 900 cu. cm. under a pressure of 420 mm. (mercury). Find the pressure if the volume becomes 800 cu. cm. and the absolute temperature 280° .

14. Sketch the graph of $y = \frac{1}{x^2}$; mark the point $P(2, \frac{1}{4})$ on it, and draw the tangent at P cutting the x -axis, Ox , at T . Find (i) the gradient of PT , (ii) the length of OT .

15. If z varies directly as x and inversely as y , complete the table :

		Values of x .				
		1	2	3	4	5
Values of y .	1					
	2			90		
	3					100

K

1. Write down the 1st and 10th terms of sequences whose n th term is

(i) $5n - 4$; (ii) $(n + 1)(n + 2)$; (iii) $n^2 - 1$.

2. Fill in the gaps in the following sequence : 2, 5, 8, , 14, 17, , 23, , 32 and write down the value (i) of the 15th term, (ii) of the n th term.

3. How many terms are there in the sequence 1, 5, 9, 13, ... 101? Find the sum of these terms.

4. A car starts from rest and is steadily accelerated in such a way that its speed after 1 second is 1 mile per hour, after 2 seconds 3 miles per hour, after 3 seconds 5 miles an hour, and so on. Find what its speed will be according to the same law after 8 seconds, and find when it will attain a speed of 20 miles an hour.

5. The horse-power of a certain type of water-wheel is given by the formula $H.P. = 0.00113QH$, where Q is quantity of water flowing in cu. feet per minute and H is the head of water in feet. Find the horse-power of a wheel of this type driven by 850 cu. feet of water per minute from a height of 2.6 feet.

6. What is the n th term of the sequence $2+1$; 2^2+2 ; 2^3+3 ; 2^4+4 ; ... ?

If the n th term of this sequence is a , the $(n+1)$ th term b , and the $(n+2)$ th term c , prove that $3b = 2a + c + 1$.

7. Find the n th term and the sum of n terms of the sequence 2, 6, 18, 54, ...

8. Find the value of $\frac{8132h^{2.5}}{n^2}$, when $h = 4.65$ and $n = 325$.

9. Prove that $\frac{101 + 103 + 105 + \dots + 199}{1 + 3 + 5 + \dots + 99} = 3$.

10. A sequence is formed whose n th term is $6n - 4$; write down the first three terms, and show that the difference between two successive terms is always 6.

11. What is the n th term of the sequence 27, 9, 3, 1, ... ? Find its value approximately when $n = 20$, using logs.

12. Find the 16th term and the sum of 16 terms of the sequence 1, $\frac{4}{3}$, $\frac{5}{3}$, 2, ...

13. Find the sum of 9 terms of a G.P. whose first two terms are 1 and $\frac{3}{5}$, using logs to get the result correct to 3 significant figures.

14. The following formula connects the penetration t inches into steel-armour plates, with the velocity v ft. per sec. of the shell :

$$t = \frac{15v}{6200 - v}$$

Find the velocity of the shell that penetrates 7". (Certificate.)

15. Find the value of $T \left(\frac{p_1}{p_2} \right)^{\frac{n-1}{n}}$ correct to 3 significant figures when $T = 147$, $p_1 = 16.5$, $p_2 = 13.8$ and $n = 1.6$.

16. Find the sum to n terms of the series $1 - \frac{1}{4} + \frac{1}{16} - \dots$. What is the "sum to infinity" of this series?

17. Evaluate 0.9 , 0.3 , $0.02\dot{4}$, $12.34\dot{5}$, $12.34\dot{5}$.

L

1. Give the general rule which includes the following :

$$(i) {}^7P_5 = \frac{7!}{2!}; \quad (ii) {}^{10}P_7 = \frac{10!}{3!}; \quad (iii) {}^{11}P_6 = \frac{11!}{5!},$$

and evaluate ${}^{20}P_2$.

2. If $p \cdot v^{1.4} = 28.7$, find v when $p = 17.3$.

3. The distance a spring stretches (*i.e.* its extension) when a weight is suspended from it varies (within certain limits) directly as the weight. The following measurements were made in an experiment; state which (if any) of them are probably incorrect, and find the probable equation between the extension l cm. and the weight w grams which produces it :

Total length of spring in cm., l - - -	21.0	22.7	26	27.7	34.2	36	39.5	41
Weight attached in gr., w - - -	0	5	15	20	35	45	50	60

4. (i) There are n circles in a plane, each lying outside the rest; what is the greatest number of lines that can be drawn, each of which touches two circles?

(ii) Two sets of 3 concentric circles are drawn in a plane, and no two circles intersect. How many lines can be drawn, each of which touches two circles?

5. The resistance R lb. to a parachute of area A sq. ft. moving v ft. per sec. varies as A if v is constant, and as v^2 if A is constant. Complete the table for R .

How does v vary if R is constant?

Values of A .

		4	8	10
Values of v .	5			
	10		1	
	15			

6. In the Morse code, letters are represented by dots or dashes or combinations of both ; how many letters can be represented by using not more than four symbols for any one letter ?

7. (i) Find the roots of $x^3 - 100x = 0$.

(ii) One root of $x^3 - 100x = 1$ is obviously nearly equal to 10 ; by putting $x = 10 + h$, where h is small, and neglecting h^2 and h^3 , find a closer approximation to this root.

8. If the diameter of a circular cylinder increases 2 per cent. and the height increases 1 per cent., find the approximate percentage increase in the volume.

9. The figure represents the graph of $y = (x-1)(x-2)$ on a certain scale. Sketch, with the same units, the graphs of

(i) $y = 1 + (x-1)(x-2)$;

(ii) $y = (x+1)(x+2)$;

(iii) $y = (x-2)(x-3)$;

(iv) $y = \frac{1}{(x-1)(x-2)}$.

Sketch also the graph of

$$y = (x-1)(x-2),$$

if the x -unit is halved and the y -unit is doubled.

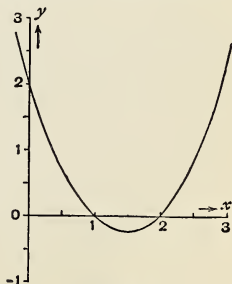


FIG. 54.

10. A teapot with a spout 2 inches long holds a pint ; how much will a teapot of the same shape with a spout 3 inches long hold ?

11. A pendulum of length l feet vibrates $\frac{108.4}{\sqrt{l}}$ times a minute. What is the reduction in the number of vibrations per hour if the length is increased, (i) from 9 to 9.1 feet, (ii) from l to $l + h$ feet, where h is small compared with l ?

If a grandfather-clock gains, would you screw the weight at the end of the pendulum up or down in regulating it ? And why ?

12. If the length of a pendulum is l feet and the time of vibration is t sec., it is found by observation that when $l = 1$, $t = 1.11$, and when $l = 2$, $t = 1.58$, and when $l = 3$, $t = 1.93$. If these obey a law of the form $t = al^n$, find a and n .

13. Draw the graph of $y = x^2 - x$. With the same axes and without any further calculation, sketch the graphs of

(i) $x = y^2 - y$; (ii) $y = x^2 + x$; (iii) $y = x^2 - x + 1$.

14. (i) If x is small, prove that $\sqrt{x^2 + 16} - \sqrt{x^2 + 9} \simeq 1 - \frac{x^2}{24}$.

(ii) If x is large, prove that $\sqrt{x^2 + 16} - \sqrt{x^2 + 9} \simeq \frac{7}{2x}$.

(iii) Hence sketch the graph of $\sqrt{x^2 + 16} - \sqrt{x^2 + 9}$.

15. The area of the wing surface of an aeroplane with load W lb. for a speed of v miles an hour varies as $\frac{W}{v^2}$. How must the area be altered approximately if W is increased by 3 per cent. and v is reduced by 2 per cent. ?

16. (i) If h is small compared with x , find an approximation for $\sqrt{x^2 + h^2} - x$.

(ii) The height, h inches, of an isosceles triangle ABC is small compared with the base BC ; the excess of $AB + AC$ over BC is d inches ; prove that the area of the triangle is approximately $\frac{h^3}{d}$.

17. The electrical resistance of a copper wire varies directly as its length and inversely as the square of the diameter of the cross-section. How is the resistance altered if the length is doubled and the diameter is halved ?

18. How many different signals can be sent, using one or more of 4 flags of different colours, (i) if different orders count as distinct signals, (ii) if the order does not matter ?

19. Motor-cars are numbered with either one or two letters (*e.g.* P ; AA ; LH) followed by any number from 1 to 9999. How many cars can be numbered differently if all letters in the alphabet are used ?

M.

1. An experiment to determine the force P lb. required to lift a weight of W lb. by a system of pulleys gave the following results :

W	5	10	15	20	30	40
P	1.23	2.15	3.06	3.98	5.82	7.64

Find the best fit equation for P in terms of W .

2. Describe in general terms the value of the function $\frac{x-4}{x-3}$, when x is (i) large and positive ; (ii) slightly greater than 4 ; (iii) slightly less than 4 ; (iv) slightly greater than 3 ; (v) slightly less than 3 ; (vi) less than -1000 .

For what range of values of x is this function negative ? Sketch its graph.

3. Simplify (i) $\frac{d}{dx}(x+2)(x-3)$; (ii) $\frac{dy}{dx}$ if $xy = 5$;
 (iii) $\int \left(x^2 - \frac{1}{x^2} \right) dx$; (iv) $\int_{-1}^{+1} (x+1)^2 dx$.

4. The following readings connect the candle-power and voltage of an incandescent lamp :

Candle-power	-	-	20.68	23.24	26.00	28.96
Voltage	-	-	94	98	102	106

If the candle-power varies as the n th power of the voltage, find n .

5. The horizontal cross-sections of the crater of a volcano are circles with their centres on a vertical line, the radii of the circles at different heights are as follows (measurements in feet) :

Height	0	10	20	30	40	50	60	70	80	90	100
Radius	0	17	24	27	30	35	41	52	65	78	94

Find its volume in cu. ft. to two significant figures.

6. A piece of wire 2 feet long is cut into two parts, one of which is bent to form a square and the other an equilateral triangle. If the sum of the areas is a minimum, find the side of the equilateral triangle.

7. The following table gives the distance, d yards, in which a train running V miles per hour can be stopped :

V	30	40	45	50	60
d	100	170	224	271	400

Plot d against V^2 , and then express d in terms of V .

8. The connection between the speed v feet per sec. of a column of water produced by a pressure of head h feet is shown in the following table :

h	4	16	25	36	64
v	15.7	32.7	39.3	49.1	63.3

Plot v against $\sqrt{h}$ and find the most probable relation expressing v in terms of h .

9. Given $y = (x - 1)^3$: (i) find the value of $\frac{dy}{dx}$ when $x = 1$; (ii) find the sign of $\frac{dy}{dx}$ when $x = 0.9$ and when $x = 1.1$; (iii) is $x = 1$ a turning point of the function $(x - 1)^3$? (iv) find the sign of $\frac{d^2y}{dx^2}$ when $x = 0.9$ and when $x = 1.1$; (v) sketch the graph of $(x - 1)^3$ from $x = 0$ to $x = 2$, and explain the geometrical interpretation of results (i) to (iv).

10. The gradient of a curve which passes through the origin is given by the equation $\frac{dy}{dx} = x^2(1 - x)$; find the area between the curve, the x -axis and the ordinate $x = 1$.

11. Sketch the graph of $(x - 2)(5 - x)$, and find the area of the portion of it which lies above the x -axis.

12. If a jet of water is projected from a $\frac{3}{4}$ -inch nozzle under a pressure of P lb. per sq. inch, the effective height, h feet, of the jet is connected with P as follows :

P	40	50	60	70	80	90	100
h	61	67	72	76	79	81	83

Plot $\frac{P}{h}$ against P , and hence find an approximate formula for h in terms of P .

13. Sketch the graph of the function $y = f(x)$ from $x = 0$ to $x = 8$, from the data in the following table :

x	0	0 to 1	1	1 to 2	2 to 4	4	4 to 6	6	6 to 8
y	0			+	—			—	
$\frac{dy}{dx}$	+	+	0	—	—	0	+	0	—

14. Interpret graphically $\int_0^1 \sqrt{1-x^2} dx$.

15. The following swimming records are taken from *Whitaker's Almanack* :

Distance in yards, d	-	150	220	300	440	500
Time in seconds, t	-	92.4	145.4	210	323	367.2

Plot $\log t$ against $\log d$, and express t in terms of d . The record for 300 metres is 230.2 sec. Is this what your formula would lead you to expect ? [1 metre = 1.094 yards.]

16. If the graph of $y = f(x)$ passes through the origin and has a maximum at $x = 1$ and a minimum at $x = 2$, but no other turning points, and if the slope at the origin is 1, find a value of $f(x)$.

17. What equation can be solved from the graphs of $xy = 4$ and $y = x^2 - x$? By *sketching* their graphs state how many real roots this equation has. Check your answer by an algebraic solution, showing that $x = 2$ is one answer.

18. The following table gives the diameters of Cornish boilers in inches for various indicated horse-powers :

Horse-power, H	-	10	20	30	40	60
Diameter, d	-	36	51	62.5	72.2	88.5

Plot $\log d$ against $\log H$, and hence express d in terms of H .

SYMBOLS.

$\delta x, \Delta x$. A change in x .

lim, Lt. See pp. 46, 47 ; Chap. III.

$n!$ $\underline{n}$. "Factorial n ." See p. 121.

$\sqrt{N}$ where $N \geq 0$. The positive (or zero) square root of N .

$-\sqrt{N}$ where $N > 0$. The negative square root of N .

$\sqrt[3]{N}$ where $N \geq 0$. The positive (or zero) cube root of N .

$\sqrt[3]{N}$ where $N < 0$. The negative cube root of N .

$|N|$. The absolute value of N . See pp. 25-26, Ex. 8 ; and Glossary.

$\approx$ is approximately equal to. See p. 78.

$\equiv$ is identically equal to. pp. 41, 60.

$\rightarrow K$ approaches the limit k .

$\rightarrow +0$ approaches the limit 0 from the positive side. See p. 24.

$\rightarrow -0$ approaches the limit 0 from the negative side. See p. 24.

$\rightarrow +\infty$ increases without limit.

$\rightarrow -\infty$ decreases without limit.

$\frac{d}{dx}$. p. 63.

$\int f(x)dx$. Pp. 88-89. $\int_a^b f(x)dx$. p. 91.

${}^nP_r$. p. 122.

${}^nC_r$. pp. 125-126.

$\log_a x$ the logarithm of x to the base a . See Glossary.

∞ . See "increases (decreases) without limit," p. 24, and "infinity" in Glossary.

Lt. $f(x)$ the limit of $f(x)$ as x approaches a .
 $x \rightarrow a$

$a:b = \frac{a}{b}$. See p. 1.

$a:b:c:d:\dots = p:q:r:s\dots$ See p. 1.

$\propto$ varies as. See p. 9.

$>$ is greater than.

$<$ is less than.

$\geq$ is equal to, or greater than.

$\leq$ is less than or equal to.

$\sqrt[n]{N}$. See in Glossary, root of a number.

$a \sim b$. The positive difference between a and b .

• Decimal point or multiplication sign. The use of $\times$ as multiplication sign is to be preferred.

GLOSSARY AND INDEX.

Abscissa. The x -coordinate of a point.

Absolute value. The absolute value of a number N is N if $N > 0$, is 0 if $N = 0$, and is $-N$ if $N < 0$. The phrase "in absolute value" is also used. p. 25.

Annuity. p. 111.

Approximation. By differentiation. p. 78.

By binomial theorem. p. 132.

Arbitrary constant. A constant whose value is as yet undetermined. p. 86.

Areas, calculation of. p. 89.

Arithmetic, mean. p. 99.

Arithmetical progression. p. 99.

Best-fit line. p. 142.

Binomial. The sum or difference of two single terms, such as $(a + b)$ or $(x - a)$.

Binomial theorem. p. 131. Chap. VIII.

Capitalization. p. 115.

Characteristic. The integral part of a logarithm.

Circle. pp. 30, 31, 59, 82.

Combination. Chap. VII.

Common difference of an A.P. p. 99.

Common ratio of a G.P. p. 102.

Complex number. Numbers that can be expressed in the form $a + bi$, where a and b are "real" and $i^2 = -1$, are called complex numbers. They are usually represented by the points of a plane. They may be classified as "real" ($b = 0$) and "imaginary" ($b \neq 0$). Imaginary numbers that can be expressed in the form $0 + bi$ are called "pure imaginaries." Examples of complex numbers: $3 + 0i$, $-\frac{4}{5}$, $\sqrt{2} + 0i$; $0 + 7i$, $-\sqrt{-1}$, $\sqrt{-5}$, the two roots of the equation $x^2 + 1 = 0$, $3 + 7i$, two of the roots of the equation $x^3 - 1 = 0$, the four roots of the equation $x^4 + 1 = 0$.

Constant. A number whose value does not change in a particular problem or part of a problem.

Cubic function of x . p. 83.

Cubic parabola. p. 84.

Decreasing function. p. 70.

Definite integral. p. 91.

Dependent variable. A variable number whose value depends upon the value of one or more independent variables; function. p. 23.

Derivative. p. 63.

Derived function. p. 63.

Differential coefficient. p. 63.

Differential equation. pp. 85-86.

Differentiation. p. 62.

Discount. p. 111.

Discriminant. p. 39.

Division by zero. See "Quotient."

Ellipse. p. 82.

Empirical formula. A formula based upon numerical results found by experiment. Chap. IX.

Equation. Straight line. p. 60. Chap. IX. Compare "linear function of x ."

Certain parabolas. pp. 25, 32 ff., 83. Compare "quadratic function of x ."

Certain circles and ellipses. pp. 30, 31, 59, 82.

Certain hyperbolas. pp. 13, 19, 23, 26, 36, 82, 84.

Equation, differential. pp. 85-86.

Extraneous root. If in the process of solving a given equation other equations are introduced, and s is a root of one of these other equations, but not a root of the given equation, s is called an extraneous root of the given equation. The name is well established but unfortunate. Rev. Ex. D.

Factorial n , where n is a positive integer, is written $\underline{n}$ or $n!$ and means the product $n(n-1)(n-2) \dots (3)(2)(1)$. p. 121.

Function. A number or quantity is said to be a function of one or more variables if its value can be determined when the values of these variables are known, and in such a case these variables are said to be independent variables. See "Dependent variable." Chap. II.

Function of x . **Linear.** $ax + b$, where a and b are constants. pp. 8-9, 60; Chap. IX.

Quadratic. $ax^2 + bx + c$, where a , b and c are constants. pp. 12 ff., 22, 32 ff., 83, 145, 146.

Cubic. $ax^3 + bx^2 + cx + d$; a , b , c , d constant. p. 82.

Power. ax^n ; a and n , constant. Chap. I., p. 84.

Growth. ab^x ; a and b , constant. p. 108.

Functional notation. p. 41.

Geometrical mean. p. 102.

Geometrical progression. p. 102.

Gradient. p. 55.

Graphical solutions. pp. 31 ff.

Growth function of x . ab^x ; a and b constant. p. 108.

Hyperbola. pp. 82, 84. Chap. I.

Imaginary number. See under "complex number" and under "real number." pp. 39, 40.

Incommensurable. Two numbers whose ratio cannot be expressed in the form $p : q$, where p and q are integers are said to be incommensurable. See "irrational number."

Increasing function. p. 70.

Indefinite integral. p. 91.

Independent variable. See "function" and "dependent variable."

Indices. Rev. Ex. A.-D.

Infinite series. pp. 105, 133 ff.

Infinity. Chaps. II. and III., note. p. 29. See in Glossary: "Varies from $-\infty$ to $+\infty$."

Integer. A whole number.

Integral. Indefinite. p. 91.

Definite. p. 91.

Integration. Chap. V.

Interpolation. When the value of a function is known for some values of the variable within a certain range, approximate values of the function may be determined for other values of the variable within this range; this process is known as interpolation. pp. 6 ff., Rev. Ex. A.-D.

Irrational number. The real numbers that are not rational are called irrational numbers. Examples: $-\sqrt[3]{7}$, $\sqrt[3]{-7}$, $3 - \pi$, $\sqrt[5]{-0.23}$, $\log_{10} 5$, $1.010010001 \dots$; $\sqrt{2}$, $10^{0.5}$, π .

Limit. Chap. III. p. 24.

Linear function of x . $ax + b$, where a and b are constants. pp. 10, 60, Chap. IX.

Logarithm. Rev. Ex. A.-D.

Logarithmic notation. Rev. Ex. A.-D.

Mantissa. The fractional part of a logarithm.

Maximum. The value of a function which is greater than all other values of the function in the immediate neighbourhood. The use of the technical word "maximum" should be avoided in cases where at the point (x_1, y_1) the function y does not change from increasing to decreasing, i.e., $\frac{dy}{dx}$ does not change from the positive sign to the negative; further, the definitions in the text are not to be regarded as applicable to graphs more complicated than Fig. 25. pp. 71, 74.

Mean. Arithmetic. p. 99.

Geometric. p. 102.

Minimum. See "maximum."

Number. See "real number" and "complex number."

Numerical value. See "absolute value." The phrases "in numerical value," "numerically equal," etc., are also used.

Ordinate. The y -coordinate of a point.

Parabola. pp. 24, 25, 33 ff., 83.

Permutation. Chap. VII.

Perpetuity. Chap. VI.

Plotting y against x . p. 10; Chap. IX.

Power function. ax^n , a and n constant. Chap. I.; p. 84.

Progression. Arithmetic. Chap. VI.

Geometric. Chap. VI.

Proportion. Chap. I.

Proportional. Chap. I.

Proportionality. Chap. I.

Pure imaginary. See under "complex number" and under "real number."

Quadratic equation in x . $ax^2 + bx + c = 0$; a, b, c , constant. pp. 13 ff., 22, 33 ff., 84, 145 ff.

Quadratic function of x . $ax^2 + bx + c$; a, b, c constant. pp. 13 ff. 22, 33 ff., 84, 145 ff.

Quotient. Given two numbers, n and d , $d \neq 0$; the quotient of n by d is q if $dq = n$. It is to be noted that $a + b$, $a - b$, $a \times b$ (or ab) all have meaning, even if one or both of the letters is 0; so also $a \div b$ has meaning if $a = 0$ and $b \neq 0$; but $a \div b$ has no meaning if $b = 0$.

Rate of change. Chaps. III. and IV.

Ratio. Chap. I.

Rational number. The integers and fractions are called rational numbers; every rational number can be expressed in the form p/q where p and q are integers; moreover, every rational number can be expressed in the form of a decimal that comes to an end or that repeats. pp. 104 ff. Examples: -23 , -6.78 , $\sqrt[3]{-8}$, 0 , 10^{-3} , $\sqrt[3]{0.001}$, $0.\dot{3}$; $2/6$, $0.\dot{9}$, 10° , 1.24563 .

Real number. The positive and negative numbers and zero are called real numbers; they are represented by the points on a linear scale. p. 39. They may be classified as (a) negative, (b) zero, and (c) positive; or classified as (1) rational, and (2) irrational; see these definitions for examples of real number.

It is essential to note that the square of every real number is positive or zero; see definition of "pure imaginary" for the kind of number whose square is negative; see also the definitions of "imaginary number" and "complex number." Although by long custom called "real," these numbers have no greater claim to "reality" than the so-called imaginary numbers.

Real root. A root that is a real number.

Real solution of an equation in x and y . A solution that is a pair of real numbers.

Repeating (recurring or circulating) decimal. p. 104.

Root of an equation. s is a root of an equation in x if, when s is substituted for x in the equation, the sides of the equation are equal. p. 39.

Root of a number. r is a k th root of N if $r^k = N$; when $k = 2$ we say a square root, when $k = 3$ we say a cube root; thus since $(+2)^2 = 4$, $+2$ is a square root of 4, and since $(-2)^3 = -8$, -2 is a cube root of -8 . For notation see list of symbols.

Sequence. A succession of numbers which obey some law; $t_1, t_2, t_3, \dots, t_n, \dots$. Chaps. VI., VII. Example: $2 \times 3^5, 3 \times 4^6, 4 \times 5^7, \dots$

Series. A succession of sums of numbers which obey some law; $t_1 + t_2 + t_3 + \dots + t_n + \dots$. Chaps. VI., VII. Example: $2 \times 3^5 + 3 \times 4^6 + 4 \times 5^7 + \dots$

Sketching a graph. p. 23. Drawing a graph which, though rough, exhibits the important features of the function.

Slope. Gradient. Chap. III.

Solution of an equation in x . See "root" of an equation.

Solution of an equation in x and y ($x = s$, $y = t$) is a solution of an equation in x and y if, when s is substituted for x and t for y in the equation, the sides of the equation are equal.

Speed. pp. 49, 52 ff., 58 ff., 82.

Straight line. Compare "linear function of x ." pp. 9, 60. Chap. IX.

Sum to infinity. The limit which the sum of n terms of a series approaches as n increases without limit. The name is unfortunate but well established. p. 105.

Turning point. p. 70.

Turning value. p. 71.

Variable. A number whose value changes in a particular problem or part of a problem.

Variation. Chap. I.

Varies from $-\infty$ to $+\infty$. Increases, running over all the real numbers, negative, zero, and positive.

Zero. See "real number" and "quotient."

TABLES

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	9	13	17	21	26	30	34	38
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	12	16	20	24	28	32	37
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	11	14	18	21	25	28	32
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	7	10	13	16	20	23	26	30
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	28
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	9	11	14	17	20	23	26
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	14	16	19	22	25
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	3	5	8	10	13	15	18	20	23
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
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33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
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41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
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LOGARITHMS

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57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
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64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
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79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
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π may be taken to be 3.1416.
 $\log \pi$ may be taken to be 0.4971.

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-87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
-88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
-89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
-90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
-91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
-92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
-93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
-94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
-95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
-96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
-97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
-98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
-99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

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10	1000	1020	1040	1061	1082	1103	1124	1145	1166	1188	2	4	6	8	10	13	15	17	19
11	1210	1232	1254	1277	1300	1323	1346	1369	1392	1416	2	5	7	9	11	14	16	18	21
12	1440	1464	1488	1513	1538	1563	1588	1613	1638	1664	2	5	7	10	12	15	17	20	22
13	1690	1716	1742	1769	1796	1823	1850	1877	1904	1932	3	5	8	11	13	16	19	22	24
14	1960	1988	2016	2045	2074	2103	2132	2161	2190	2220	3	6	9	12	14	17	20	23	26
15	2250	2280	2310	2341	2372	2403	2434	2465	2496	2528	3	6	9	12	15	19	22	25	28
16	2560	2592	2624	2657	2690	2723	2756	2789	2822	2856	3	7	10	13	16	20	23	26	30
17	2890	2924	2958	2993	3028	3063	3098	3133	3168	3204	3	7	10	14	17	21	24	28	31
18	3240	3276	3312	3349	3386	3423	3460	3497	3534	3572	4	7	11	15	18	22	26	30	33
19	3610	3648	3686	3725	3764	3803	3842	3881	3920	3960	4	8	12	16	19	23	27	31	35
20	4000	4040	4080	4121	4162	4203	4244	4285	4326	4368	4	8	12	16	20	25	29	33	37
21	4410	4452	4494	4537	4580	4623	4666	4709	4752	4796	4	9	13	17	21	26	30	34	39
22	4840	4884	4928	4973	5018	5063	5108	5153	5198	5244	4	9	13	18	22	27	31	36	40
23	5290	5336	5382	5429	5476	5523	5570	5617	5664	5712	5	9	14	19	23	28	33	38	42
24	5760	5808	5856	5905	5954	6003	6052	6101	6150	6200	5	10	15	20	24	29	34	39	44
25	6250	6300	6350	6401	6452	6503	6554	6605	6656	6708	5	10	15	20	25	31	36	41	46
26	6760	6812	6864	6917	6970	7023	7076	7129	7182	7236	5	11	16	21	26	32	37	42	48
27	7290	7344	7398	7453	7508	7563	7618	7673	7728	7784	5	11	16	22	27	33	38	44	49
28	7840	7896	7952	8009	8066	8123	8180	8237	8294	8352	6	11	17	23	28	34	40	46	51
29	8410	8468	8526	8585	8644	8703	8762	8821	8880	8940	6	12	18	24	29	35	41	47	53
30	9000	9060	9120	9181	9242	9303	9364	9425	9486	9548	6	12	18	24	30	37	43	49	55
31	9610	9672	9734	9797	9860	9923	9986				6	13	19	25	31	38	44	50	57
32	1024	1030	1037	1043	1050	1056	1063	1069	1076	1082	1	1	2	3	3	4	5	5	6
33	1089	1096	1102	1109	1116	1122	1129	1136	1142	1149	1	1	2	3	3	4	5	5	6
34	1156	1163	1170	1176	1183	1190	1197	1204	1211	1218	1	1	2	3	3	4	5	6	6
35	1225	1232	1239	1246	1253	1260	1267	1274	1282	1289	1	1	2	3	4	4	5	6	6
36	1296	1303	1310	1318	1325	1332	1340	1347	1354	1362	1	1	2	3	4	4	5	6	7
37	1369	1376	1384	1391	1399	1406	1414	1421	1429	1436	1	2	2	3	4	5	5	6	7
38	1444	1452	1459	1467	1475	1482	1490	1498	1505	1513	1	2	2	3	4	5	5	6	7
39	1521	1529	1537	1544	1552	1560	1568	1576	1584	1592	1	2	2	3	4	5	6	6	7
40	1600	1608	1616	1624	1632	1640	1648	1656	1665	1673	1	2	2	3	4	5	6	6	7
41	1681	1689	1697	1706	1714	1722	1731	1739	1747	1756	1	2	2	3	4	5	6	7	7
42	1764	1772	1781	1789	1798	1806	1815	1823	1832	1840	1	2	3	3	4	5	6	7	8
43	1849	1858	1866	1875	1884	1892	1901	1910	1918	1927	1	2	3	3	4	5	6	7	8
44	1936	1945	1954	1962	1971	1980	1989	1998	2007	2016	1	2	3	4	4	5	6	7	8
45	2025	2034	2043	2052	2061	2070	2079	2088	2098	2107	1	2	3	4	5	5	6	7	8
46	2116	2125	2134	2144	2153	2162	2172	2181	2190	2200	1	2	3	4	5	6	7	7	8
47	2209	2218	2228	2237	2247	2256	2266	2275	2285	2294	1	2	3	4	5	6	7	8	9
48	2304	2314	2323	2333	2343	2352	2362	2372	2381	2391	1	2	3	4	5	6	7	8	9
49	2401	2411	2421	2430	2440	2450	2460	2470	2480	2490	1	2	3	4	5	6	7	8	9
50	2500	2510	2520	2530	2540	2550	2560	2570	2581	2591	1	2	3	4	5	6	7	8	9
51	2601	2611	2621	2632	2642	2652	2663	2673	2683	2694	1	2	3	4	5	6	7	8	9
52	2704	2714	2725	2735	2746	2756	2767	2777	2788	2798	1	2	3	4	5	6	7	8	9
53	2809	2820	2830	2841	2852	2862	2873	2884	2894	2905	1	2	3	4	5	6	7	9	10
54	2916	2927	2938	2948	2959	2970	2981	2992	3003	3014	1	2	3	4	5	7	8	9	10

Find the position of the decimal point by inspection.

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55	3025	3036	3047	3058	3069	3080	3091	3102	3114	3125	1	2	3	4	5	6	7	8	9
56	3136	3147	3158	3170	3181	3192	3204	3215	3226	3238	1	2	3	4	5	6	7	8	9
57	3249	3260	3272	3283	3295	3306	3318	3329	3341	3352	1	2	3	4	5	6	7	8	9
58	3364	3376	3387	3399	3411	3422	3434	3446	3457	3469	1	2	4	5	6	7	8	9	11
59	3481	3493	3505	3516	3528	3540	3552	3564	3576	3588	1	2	4	5	6	7	8	10	11
60	3600	3612	3624	3636	3648	3660	3672	3684	3697	3709	1	2	4	5	6	7	8	10	11
61	3721	3733	3745	3758	3770	3782	3795	3807	3819	3832	1	2	4	5	6	7	9	10	11
62	3844	3856	3869	3881	3894	3906	3919	3931	3944	3956	1	3	4	5	6	7	9	10	11
63	3969	3982	3994	4007	4020	4032	4045	4058	4070	4083	1	3	4	5	6	8	9	10	11
64	4096	4109	4122	4134	4147	4160	4173	4186	4199	4212	1	3	4	5	6	8	9	10	12
65	4225	4238	4251	4264	4277	4290	4303	4316	4330	4343	1	3	4	5	7	8	9	10	12
66	4356	4369	4382	4396	4409	4422	4436	4449	4462	4476	1	3	4	5	7	8	9	11	12
67	4489	4502	4516	4529	4543	4556	4570	4583	4597	4610	1	3	4	5	7	8	9	11	12
68	4624	4638	4651	4665	4679	4692	4706	4720	4733	4747	1	3	4	5	7	8	10	11	12
69	4761	4775	4789	4802	4816	4830	4844	4858	4872	4886	1	3	4	6	7	8	10	11	13
70	4900	4914	4928	4942	4956	4970	4984	4998	5013	5027	1	3	4	6	7	8	10	11	13
71	5041	5055	5069	5084	5098	5112	5127	5141	5155	5170	1	3	4	6	7	9	10	11	13
72	5184	5198	5213	5227	5242	5256	5271	5285	5300	5314	1	3	4	6	7	9	10	12	13
73	5329	5344	5358	5373	5388	5402	5417	5432	5446	5461	1	3	4	6	7	9	10	12	13
74	5476	5491	5506	5520	5535	5550	5565	5580	5595	5610	1	3	4	6	7	9	10	12	13
75	5625	5640	5655	5670	5685	5700	5715	5730	5746	5761	2	3	5	6	8	9	11	12	14
76	5776	5791	5806	5822	5837	5852	5868	5883	5898	5914	2	3	5	6	8	9	11	12	14
77	5929	5944	5960	5975	5991	6006	6022	6037	6053	6068	2	3	5	6	8	9	11	12	14
78	6084	6100	6116	6131	6147	6162	6178	6194	6209	6225	2	3	5	6	8	9	11	12	14
79	6241	6257	6273	6288	6304	6320	6336	6352	6368	6384	2	3	5	6	8	10	11	13	14
80	6400	6416	6432	6448	6464	6480	6496	6512	6529	6545	2	3	5	6	8	10	11	13	14
81	6561	6577	6593	6610	6626	6642	6659	6675	6691	6708	2	3	5	7	8	10	11	13	15
82	6724	6740	6757	6773	6790	6806	6823	6839	6856	6872	2	3	5	7	8	10	12	13	15
83	6889	6906	6922	6939	6956	6972	6989	7006	7022	7039	2	3	5	7	8	10	12	13	15
84	7056	7073	7090	7106	7123	7140	7157	7174	7191	7208	2	3	5	7	8	10	12	14	15
85	7225	7242	7259	7276	7293	7310	7327	7344	7362	7379	2	3	5	7	9	10	12	14	15
86	7396	7413	7430	7448	7465	7482	7500	7517	7534	7552	2	3	5	7	9	10	12	14	16
87	7569	7586	7604	7621	7639	7656	7674	7691	7709	7726	2	4	5	7	9	10	12	14	16
88	7744	7762	7779	7797	7815	7832	7850	7868	7885	7903	2	4	5	7	9	11	12	14	16
89	7921	7939	7957	7974	7992	8010	8028	8046	8064	8082	2	4	5	7	9	11	13	14	16
90	8100	8118	8136	8154	8172	8190	8208	8226	8245	8263	2	4	5	7	9	11	13	14	16
91	8281	8299	8317	8336	8354	8372	8391	8409	8427	8446	2	4	5	7	9	11	13	15	16
92	8464	8482	8501	8519	8538	8556	8575	8593	8612	8630	2	4	6	7	9	11	13	15	17
93	8649	8668	8686	8705	8724	8742	8761	8780	8798	8817	2	4	6	7	9	11	13	15	17
94	8836	8855	8874	8892	8911	8930	8949	8968	8987	9006	2	4	6	8	9	11	13	15	17
95	9025	9044	9063	9082	9101	9120	9139	9158	9177	9197	2	4	6	8	10	11	13	15	17
96	9216	9235	9254	9274	9293	9312	9332	9351	9370	9390	2	4	6	8	10	12	14	15	17
97	9409	9428	9448	9467	9487	9506	9526	9545	9565	9584	2	4	6	8	10	12	14	16	18
98	9604	9624	9643	9663	9683	9702	9722	9742	9761	9781	2	4	6	8	10	12	14	16	18
99	9801	9821	9841	9860	9880	9900	9920	9940	9960	9980	2	4	6	8	10	12	14	16	18

Find the position of the decimal point by inspection.

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10	1000	1005	1010	1015	1020	1025	1030	1034	1039	1044	0	1	1	2	2	3	3	4	4
11	1049	1054	1058	1063	1068	1072	1077	1082	1086	1091	0	1	1	2	2	3	3	4	4
12	1095	1100	1105	1109	1114	1118	1122	1127	1131	1136	0	1	1	2	2	3	3	4	4
13	1140	1145	1149	1153	1158	1162	1166	1170	1175	1179	0	1	1	2	2	3	3	4	4
14	1183	1187	1192	1196	1200	1204	1208	1212	1217	1221	0	1	1	2	2	2	3	3	4
15	1225	1229	1233	1237	1241	1245	1249	1253	1257	1261	0	1	1	2	2	2	3	3	4
16	1265	1269	1273	1277	1281	1285	1288	1292	1296	1300	0	1	1	2	2	2	3	3	4
17	1304	1308	1311	1315	1319	1323	1327	1330	1334	1338	0	1	1	2	2	2	3	3	3
18	1342	1345	1349	1353	1356	1360	1364	1367	1371	1375	0	1	1	1	2	2	3	3	3
19	1378	1382	1386	1389	1393	1396	1400	1404	1407	1411	0	1	1	1	2	2	3	3	3
20	1414	1418	1421	1425	1428	1432	1435	1439	1442	1446	0	1	1	1	2	2	2	3	3
21	1449	1453	1456	1459	1463	1466	1470	1473	1476	1480	0	1	1	1	2	2	2	3	3
22	1483	1487	1490	1493	1497	1500	1503	1507	1510	1513	0	1	1	1	2	2	2	3	3
23	1517	1520	1523	1526	1530	1533	1536	1539	1543	1546	0	1	1	1	2	2	2	3	3
24	1549	1552	1556	1559	1562	1565	1568	1572	1575	1578	0	1	1	1	2	2	2	3	3
25	1581	1584	1587	1591	1594	1597	1600	1603	1606	1609	0	1	1	1	2	2	2	3	3
26	1612	1616	1619	1622	1625	1628	1631	1634	1637	1640	0	1	1	1	2	2	2	2	3
27	1643	1646	1649	1652	1655	1658	1661	1664	1667	1670	0	1	1	1	2	2	2	2	3
28	1673	1676	1679	1682	1685	1688	1691	1694	1697	1700	0	1	1	1	1	2	2	2	3
29	1703	1706	1709	1712	1715	1718	1720	1723	1726	1729	0	1	1	1	1	2	2	2	3
30	1732	1735	1738	1741	1744	1746	1749	1752	1755	1758	0	1	1	1	1	2	2	2	3
31	1761	1764	1766	1769	1772	1775	1778	1780	1783	1786	0	1	1	1	1	2	2	2	3
32	1789	1792	1794	1797	1800	1803	1806	1808	1811	1814	0	1	1	1	1	2	2	2	2
33	1817	1819	1822	1825	1828	1830	1833	1836	1838	1841	0	1	1	1	1	2	2	2	2
34	1844	1847	1849	1852	1855	1857	1860	1863	1865	1868	0	1	1	1	1	2	2	2	2
35	1871	1873	1876	1879	1881	1884	1887	1889	1892	1895	0	1	1	1	1	2	2	2	2
36	1897	1900	1903	1905	1908	1910	1913	1916	1918	1921	0	1	1	1	1	2	2	2	2
37	1924	1926	1929	1931	1934	1936	1939	1942	1944	1947	0	1	1	1	1	2	2	2	2
38	1949	1952	1954	1957	1960	1962	1965	1967	1970	1972	0	1	1	1	1	2	2	2	2
39	1975	1977	1980	1982	1985	1987	1990	1992	1995	1997	0	1	1	1	1	2	2	2	2
40	2000	2002	2005	2007	2010	2012	2015	2017	2020	2022	0	0	1	1	1	1	2	2	2
41	2025	2027	2030	2032	2035	2037	2040	2042	2045	2047	0	0	1	1	1	1	2	2	2
42	2049	2052	2054	2057	2059	2062	2064	2066	2069	2071	0	0	1	1	1	1	2	2	2
43	2074	2076	2078	2081	2083	2086	2088	2090	2093	2095	0	0	1	1	1	1	2	2	2
44	2098	2100	2102	2105	2107	2110	2112	2114	2117	2119	0	0	1	1	1	1	2	2	2
45	2121	2124	2126	2128	2131	2133	2135	2138	2140	2142	0	0	1	1	1	1	2	2	2
46	2145	2147	2149	2152	2154	2156	2159	2161	2163	2166	0	0	1	1	1	1	2	2	2
47	2168	2170	2173	2175	2177	2179	2182	2184	2186	2189	0	0	1	1	1	1	2	2	2
48	2191	2193	2195	2198	2200	2202	2205	2207	2209	2211	0	0	1	1	1	1	2	2	2
49	2214	2216	2218	2220	2223	2225	2227	2229	2232	2234	0	0	1	1	1	1	2	2	2
50	2236	2238	2241	2243	2245	2247	2249	2252	2254	2256	0	0	1	1	1	1	2	2	2
51	2258	2261	2263	2265	2267	2269	2272	2274	2276	2278	0	0	1	1	1	1	2	2	2
52	2280	2283	2285	2287	2289	2291	2293	2296	2298	2300	0	0	1	1	1	1	2	2	2
53	2302	2304	2307	2309	2311	2313	2315	2317	2319	2322	0	0	1	1	1	1	2	2	2
54	2324	2326	2328	2330	2332	2335	2337	2339	2341	2343	0	0	1	1	1	1	2	2	2

Find the first significant figure and the position of the decimal point by inspection.

SQUARE ROOTS

201

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
10	3162	3178	3194	3209	3225	3240	3256	3271	3286	3302	2 3 5	6 8 9	11 12 14
11	3317	3332	3347	3362	3376	3391	3406	3421	3435	3450	1 3 4	6 7 9	10 12 13
12	3464	3479	3493	3507	3521	3536	3550	3564	3578	3592	1 3 4	6 7 8	10 11 13
13	3606	3619	3633	3647	3661	3674	3688	3701	3715	3728	1 3 4	5 7 8	10 11 12
14	3742	3755	3768	3782	3795	3808	3821	3834	3847	3860	1 3 4	5 7 8	9 11 12
15	3873	3886	3899	3912	3924	3937	3950	3962	3975	3987	1 3 4	5 6 8	9 10 11
16	4000	4012	4025	4037	4050	4062	4074	4087	4099	4111	1 2 4	5 6 7	9 10 11
17	4123	4135	4147	4159	4171	4183	4195	4207	4219	4231	1 2 4	5 6 7	8 10 11
18	4243	4254	4266	4278	4290	4301	4313	4324	4336	4347	1 2 3	5 6 7	8 9 10
19	4359	4370	4382	4393	4405	4416	4427	4438	4450	4461	1 2 3	5 6 7	8 9 10
20	4472	4483	4494	4506	4517	4528	4539	4550	4561	4572	1 2 3	4 6 7	8 9 10
21	4583	4593	4604	4615	4626	4637	4648	4658	4669	4680	1 2 3	4 5 6	8 9 10
22	4690	4701	4712	4722	4733	4743	4754	4764	4775	4785	1 2 3	4 5 6	7 8 9
23	4796	4806	4817	4827	4837	4848	4858	4868	4879	4889	1 2 3	4 5 6	7 8 9
24	4899	4909	4919	4930	4940	4950	4960	4970	4980	4990	1 2 3	4 5 6	7 8 9
25	5000	5010	5020	5030	5040	5050	5060	5070	5079	5089	1 2 3	4 5 6	7 8 9
26	5099	5109	5119	5128	5138	5148	5158	5167	5177	5187	1 2 3	4 5 6	7 8 9
27	5196	5206	5215	5225	5235	5244	5254	5263	5273	5282	1 2 3	4 5 6	7 8 9
28	5292	5301	5310	5320	5329	5339	5348	5357	5367	5376	1 2 3	4 5 6	7 7 8
29	5385	5394	5404	5413	5422	5431	5441	5450	5459	5468	1 2 3	4 5 5	6 7 8
30	5477	5486	5495	5505	5514	5523	5532	5541	5550	5559	1 2 3	4 4 5	6 7 8
31	5568	5577	5586	5595	5604	5612	5621	5630	5639	5648	1 2 3	3 4 5	6 7 8
32	5657	5666	5675	5683	5692	5701	5710	5718	5727	5736	1 2 3	3 4 5	6 7 8
33	5745	5753	5762	5771	5779	5788	5797	5805	5814	5822	1 2 3	3 4 5	6 7 8
34	5831	5840	5848	5857	5865	5874	5882	5891	5899	5908	1 2 3	3 4 5	6 7 8
35	5916	5925	5933	5941	5950	5958	5967	5975	5983	5992	1 2 2	3 4 5	6 7 8
36	6000	6008	6017	6025	6033	6042	6050	6058	6066	6075	1 2 2	3 4 5	6 7 7
37	6083	6091	6099	6107	6116	6124	6132	6140	6148	6156	1 2 2	3 4 5	6 7 7
38	6164	6173	6181	6189	6197	6205	6213	6221	6229	6237	1 2 2	3 4 5	6 6 7
39	6245	6253	6261	6269	6277	6285	6293	6301	6309	6317	1 2 2	3 4 5	6 6 7
40	6325	6332	6340	6348	6356	6364	6372	6380	6387	6395	1 2 2	3 4 5	6 6 7
41	6403	6411	6419	6427	6434	6442	6450	6458	6465	6473	1 2 2	3 4 5	5 6 7
42	6481	6488	6496	6504	6512	6519	6527	6535	6542	6550	1 2 2	3 4 5	5 6 7
43	6557	6565	6573	6580	6588	6595	6603	6611	6618	6626	1 2 2	3 4 5	5 6 7
44	6633	6641	6648	6656	6663	6671	6678	6686	6693	6701	1 2 2	3 4 5	5 6 7
45	6708	6716	6723	6731	6738	6745	6753	6760	6768	6775	1 1 2	3 4 4	5 6 7
46	6782	6790	6797	6804	6812	6819	6826	6834	6841	6848	1 1 2	3 4 4	5 6 7
47	6856	6863	6870	6877	6885	6892	6899	6907	6914	6921	1 1 2	3 4 4	5 6 7
48	6928	6935	6943	6950	6957	6964	6971	6979	6986	6993	1 1 2	3 4 4	5 6 6
49	7000	7007	7014	7021	7029	7036	7043	7050	7057	7064	1 1 2	3 4 4	5 6 6
50	7071	7078	7085	7092	7099	7106	7113	7120	7127	7134	1 1 2	3 4 4	5 6 6
51	7141	7148	7155	7162	7169	7176	7183	7190	7197	7204	1 1 2	3 4 4	5 6 6
52	7211	7218	7225	7232	7239	7246	7253	7259	7266	7273	1 1 2	3 3 4	5 6 6
53	7280	7287	7294	7301	7308	7314	7321	7328	7335	7342	1 1 2	3 3 4	5 5 6
54	7348	7355	7362	7369	7376	7382	7389	7396	7403	7409	1 1 2	3 3 4	5 5 6

Find the first significant figure and the position of the decimal point by inspection.

<i>n</i>	2½%	3%	3½%	4%	4½%	5%	<i>n</i>
1	1'02500	1'03000	1'03500	1'04000	1'04500	1'05000	1
2	1'05063	1'06090	1'07123	1'08160	1'09203	1'10250	2
3	1'07689	1'09273	1'10872	1'12486	1'14117	1'15763	3
4	1'10381	1'12551	1'14752	1'16986	1'19252	1'21551	4
5	1'13141	1'15927	1'18769	1'21665	1'24618	1'27628	5
6	1'15969	1'19405	1'22926	1'26532	1'30226	1'34010	6
7	1'18869	1'22987	1'27228	1'31593	1'36086	1'40710	7
8	1'21840	1'26677	1'31681	1'36857	1'42210	1'47746	8
9	1'24886	1'30477	1'36290	1'42331	1'48610	1'55133	9
10	1'28008	1'34392	1'41060	1'48024	1'55297	1'62889	10
11	1'31209	1'38423	1'45997	1'53945	1'62285	1'71034	11
12	1'34489	1'42576	1'51107	1'60103	1'69588	1'79586	12
13	1'37851	1'46853	1'56396	1'66507	1'77220	1'88565	13
14	1'41297	1'51259	1'61869	1'73168	1'85194	1'97993	14
15	1'44830	1'55797	1'67535	1'80094	1'93528	2'07893	15
16	1'48451	1'60471	1'73399	1'87298	2'02237	2'18287	16
17	1'52162	1'65285	1'79468	1'94790	2'11338	2'29202	17
18	1'55966	1'70243	1'85749	2'02582	2'20848	2'40662	18
19	1'59865	1'75351	1'92250	2'10685	2'30786	2'52695	19
20	1'63862	1'80611	1'98979	2'19112	2'41171	2'65330	20
21	1'67958	1'86029	2'05943	2'27877	2'52024	2'78596	21
22	1'72157	1'91610	2'13151	2'36992	2'63365	2'92526	22
23	1'76461	1'97359	2'20611	2'46472	2'75217	3'07152	23
24	1'80873	2'03279	2'28333	2'56330	2'87601	3'22510	24
25	1'85394	2'09378	2'36324	2'66584	3'00543	3'38635	25
26	1'90029	2'15659	2'44596	2'77247	3'14068	3'55567	26
27	1'94780	2'22129	2'53157	2'88337	3'28201	3'73346	27
28	1'99650	2'28793	2'62017	2'99870	3'42970	3'92013	28
29	2'04641	2'35657	2'71188	3'11865	3'58404	4'11614	29
30	2'09757	2'42726	2'80679	3'24340	3'74532	4'32194	30
31	2'15001	2'50008	2'90503	3'37313	3'91386	4'53804	31
32	2'20376	2'57508	3'00671	3'50806	4'08998	4'76494	32
33	2'25885	2'65234	3'11194	3'64838	4'27403	5'00319	33
34	2'31532	2'73191	3'22086	3'79432	4'46636	5'25335	34
35	2'37321	2'81386	3'33359	3'94609	4'66735	5'51602	35
36	2'43254	2'89828	3'45027	4'10393	4'87738	5'79182	36
37	2'49335	2'98523	3'57103	4'26809	5'09686	6'08141	37
38	2'55568	3'07478	3'69601	4'43881	5'32622	6'38548	38
39	2'61957	3'16703	3'82537	4'61637	5'56590	6'70475	39
40	2'68506	3'26204	3'95926	4'80102	5'81636	7'03999	40

These tables are used by permission of the University of Toronto Press.

n	2½%	3%	3½%	4%	4½%	5%	n
1	·97561	·97087	·96618	·96154	·95694	·95238	1
2	·95181	·94260	·93351	·92456	·91573	·90703	2
3	·92860	·91514	·90194	·88900	·87630	·86384	3
4	·90595	·88849	·87144	·85480	·83856	·82270	4
5	·88385	·86261	·84197	·82193	·80245	·78353	5
6	·86230	·83748	·81350	·79031	·76790	·74622	6
7	·84127	·81309	·78599	·75992	·73483	·71068	7
8	·82075	·78941	·75941	·73069	·70319	·67684	8
9	·80073	·76642	·73373	·70259	·67290	·64461	9
10	·78120	·74409	·70892	·67556	·64393	·61391	10
11	·76214	·72242	·68495	·64958	·61620	·58468	11
12	·74356	·70138	·66178	·62460	·58966	·55684	12
13	·72542	·68095	·63940	·60057	·56427	·53032	13
14	·70773	·66112	·61778	·57748	·53997	·50507	14
15	·69047	·64186	·59689	·55526	·51672	·48102	15
16	·67362	·62317	·57671	·53391	·49447	·45811	16
17	·65720	·60502	·55720	·51337	·47318	·43630	17
18	·64117	·58739	·53836	·49363	·45280	·41552	18
19	·62553	·57029	·52016	·47464	·43330	·39573	19
20	·61027	·55368	·50257	·45639	·41464	·37689	20
21	·59539	·53755	·48557	·43883	·39679	·35894	21
22	·58086	·52189	·46915	·42196	·37970	·34185	22
23	·56670	·50669	·45329	·40573	·36335	·32557	23
24	·55288	·49193	·43796	·39012	·34770	·31007	24
25	·53939	·47761	·42315	·37512	·33273	·29530	25
26	·52623	·46369	·40884	·36069	·31840	·28124	26
27	·51340	·45019	·39501	·34682	·30469	·26785	27
28	·50088	·43708	·38165	·33348	·29157	·25509	28
29	·48866	·42435	·36875	·32065	·27902	·24295	29
30	·47674	·41199	·35628	·30832	·26700	·23138	30
31	·46511	·39999	·34423	·29646	·25550	·22036	31
32	·45377	·38834	·33259	·28506	·24450	·20987	32
33	·44270	·37703	·32134	·27409	·23397	·19987	33
34	·43191	·36604	·31048	·26355	·22390	·19035	34
35	·42137	·35538	·29998	·25342	·21425	·18129	35
36	·41109	·34593	·28983	·24367	·20503	·17266	36
37	·40107	·33498	·28003	·23430	·19620	·16444	37
38	·39128	·32523	·27056	·22529	·18775	·15661	38
39	·38174	·31575	·26141	·21662	·17967	·14915	39
40	·37243	·30656	·25257	·20829	·17193	·14205	40

ANSWERS.

EXERCISE I. a. (p. 2.)

1. $8 : 3, 2 : 3, 1 : b^2, 1 : x, 5x : 4y, 1 : 4, 1000 : 1, (a + b) : a, 3a : b, 100x : y.$
2. $9 : 2.$
3. $-4.$
5. $\frac{a}{c} = \frac{d}{b}, \frac{p}{q} = \frac{r}{p}, \frac{a+b}{p} = \frac{e+f}{c+d}, \frac{x}{p} \cdot \frac{y}{q} \cdot \frac{z}{r} = 1, \frac{a}{d} \cdot \frac{b}{e} \cdot \frac{c}{1} = 1.$
6. $\frac{bx}{a}.$
7. $\frac{10}{11}.$
8. $\frac{100x}{100+x}.$
9. $\frac{x+y}{x}.$
10. $\frac{1}{2}.$
11. $\frac{4a}{5b}.$
12. $\$ \frac{ap}{a+b}, \frac{bp}{a+b}.$
13. $aq : bp.$
14. $28, 20; \frac{pc}{p-q}, \frac{qc}{p-q}.$
15. $\frac{5x}{3}.$ ✓
16. $8 : 3.$
17. $6 : 1.$
18. $\left[\frac{bx}{a} \right] \frac{100(b-a)}{a}.$
19. Greater.
20. Less.
21. $\frac{1}{2}.$
22. $8 : 6 : 9.$
23. $\frac{l(a-b)}{2(a+b)} \text{ ft.}$
24. $\frac{px}{qy}, \frac{px}{q(x+y)}.$
25. $\frac{6}{5} \text{ or } -\frac{2}{11}.$
26. $3 : 2.$
27. $5 : 4.$
28. $\frac{ab}{a+b}.$
29. $\left[\frac{55}{55} \right].$
30. $1.$
31. $3 : 2.$
32. $1 : 2.$
33. $27 : 8, 75 : 4.$
34. Increased in ratio $3 : 2.$
35. $\frac{a+x}{b+x} < \frac{a}{b} < \frac{a-x}{b-x}.$
36. $\frac{b-a}{a}.$
37. $\frac{(a+b)(qc-pd)}{qa-pb}.$
38. 2000.
39. $\frac{x}{y}.$
40. $8 : 5.$

I. b. (p. 6.)

1. $6, \frac{c^2d}{a}, x.$
2. $4\frac{1}{2}, \frac{bc^2}{a}, xy^2.$
3. $\pm 6, \pm a^2b^2, \pm \sqrt{(ab)}.$
4. 11, 5, 4.
5. 1, $14 : 21 : 6.$
6. $29 : 9 : 6.$
7. $5 : 7 : 4.$
10. $b + d + f, a - c + e, 10b - 7d + 2f.$
15. 5, 20 (19), 8, 76; ($z = x^2y + 2x + y$).

I. c. (p. 12.)

1. Yes, doubles y , doubles x , $y = 88x$.
2. No, halves y , doubles x , $y = \frac{480}{x}$.
3. Trebles y , halves x , line through origin.
4. Halves y , divides x by 10, curve, area is constant.
5. $y \propto \frac{1}{x}$; $x = 6, 3, 2, 1.5, 1.2$; $\frac{1}{x} = \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}.$
6. Halves x , divides y by 16, curve.
8. $y \propto \frac{1}{x^2}.$
9. $[3 : 1]$ 10 : 3, 18 : 5. no.
10. $y = 4x, y = 2x^2, xy = 10, y = \sqrt{x}.$
11. $[A \propto h]$ 9, 12, 15, 24, 30.
12. 0, 3, 12, 27, 75. 300.
13. 8, 2, 0.8, 0.4.
14. 2.6, 20.8, 166.4.
15. 0, 2, 2.83, 4, 10, 20.
16. $x^2 = 5y.$
17. $xy = 8.$
19. Increases in ratio 512 : 1, $V^2 \propto S^3$
20. Yes, $P = 0.00492v^2.$
21. 7.78 mi., $N = 1.23\sqrt{h}.$
22. $[6\frac{3}{4}]$ gallons.
23. $[\pounds 4 \text{ } 13\text{s. } 9\text{d.}]$
24. $1.41 : 1, \underline{1.26 : 1},$
25. .0934 oz.
26. $1.755x$ lbs.
27. $1.59 : 1.$
28. 1 sec., $t = 0.2\sqrt{l}.$

I. d. (p. 18.)

1. $V \propto h, V \propto r^2, V \propto r^2h, V = \frac{1}{3}\pi r^2h.$
2. $W \propto b, W \propto d^2, W \propto \frac{1}{l}, W \propto \frac{bd^2}{l}, W = \frac{3bd^2}{16l}.$
3. $T \propto \frac{1}{l}, T \propto v^2, T \propto \frac{v^2}{l}, T = \frac{3v^2}{32l}, 28.3 \text{ ft. per sec. } v \propto \sqrt{L}.$

4. $T \propto \ln^2 W$, $T = \frac{\pi^2 \ln^2 W}{8}$, $n \propto \frac{1}{\sqrt{l}}$, $l \propto \frac{T}{n^2}$, doubles T .
5. Top row, 270, 315, 360; second, 216, 252, 288; third, 180, 210, 240.
6. Top row, 3, 6, 9; second, 12, 24, 36; third, 27, 54, 81.
7. $\frac{24}{xy^2}$. 8. $S \propto lr$, $p \propto \frac{T}{V}$, $t \propto \frac{d}{v}$, $p \propto \frac{lx^2}{d^3}$, $t \propto \sqrt{\left(\frac{l^3}{dp}\right)}$.
9. $d = 0.8 \frac{Wv^2}{R}$. 10. $d = \lambda \cdot \sqrt[4]{\left(\frac{i^2 H}{P}\right)}$, reduced 27 per cent.
11. $H = 0.00504 d^{\frac{2}{3}} v^3$. 12. $C \propto T v^3$, $C = \frac{d^3}{50 T^2}$.
13. Decreases in ratio 1 : 6.

II. a. (p. 25.)

1. 1, 3; $x > 3$ and $x > 1$. 2. No. 3. 2, -3; $2 > x > -3$.
4. 9; 4, -2; $-\infty$, $-\infty$. 7. $x > 10$ or $x < -10$; $0.1 > x > -0.1$.
8. Small, positive; small, negative; large, positive; large, negative.
10. $\sqrt{(25 - x^2)}$; circle, centre O ; $\pm \sqrt{(25 - x^2)}$.
14. Yes, 2.01, 1.999, 1; $2 > x > 1$; no, 1002, -998.
15. 1, 3, 4; $x > 4$ and $3 > x > 1$; 378, -12, -2002; yes, *e.g.*, 104.
16. Yes, *e.g.*, 2.999 and 3.001; no; 1, 5; 0.3, 0.3 per cent.
18. 98, -102, -6, -201, 199; small, 0.000001; -0.000001; $3 > x > 2$ and $1 > x$.

II. b. (p. 29.)

1. $y = \frac{1}{2}(x + 4)$. 2. $y = \sqrt{(8x - x^2)}$. 3. $y = \sqrt{(8x - 16)}$.
4. $y = \sqrt{(6 - x^2)}$. 5. $z = \sqrt{(4x - x^2)}$; $y = \sqrt{\frac{x^3}{4 - x}}$.
6. $y = \frac{4}{3}\sqrt{(9 - x^2)}$. 7. $z(1 - x)(3x - 1)$. 8. $y = \frac{abx}{x^2 + b^2}$.
9. $y = \frac{d}{x}\sqrt{(d^2 - x^2)}$. 10. $x = \frac{c(by + az)}{(a + b)(b + c)}$. 11. $V = 0.0293 A^{1.5}$.
12. $\frac{1}{2}(x^3 \sim 1)$. 13. $\frac{\pi x^2}{192}\sqrt{(16 - x^2)}$.
14. $(3 - x)(x - 1)$. 15. $(x - 1)^2 + 2$.

II. c. (p. 33.)

1. 2.5.
2. $7.5, \frac{1}{2}x \sqrt{10x - x^2}$.
3. 2.94.
4. $A = \frac{10x(x - 10)}{r - 20}, 5.86$.
5. $A = x^2 + \frac{800}{x}, 3.68, 10.1$ or 1.54 . 6. $V = 4x(12 - x)^2, 1024$ cu. in.
7. $V = \frac{1}{2}x^2(15 - 4x), 15.6$. 8. $V = 2\pi r^2(6 - r), 201$.
9. $-4, 1, 4, 5, 4, 1, -4$; (i) 3.24 or -1.24 ; (ii) $3.24 > x > -1.24$; (iii) $5, x = 1$; (iv) 2.87 or -0.87 ; (v) $2, 2.73$ or -0.73 ; (vi) 3.45 or -1.45 .
10. $4.2, 1.8, 0.2, -0.6, -0.6, 0.2, 1.8, 4.2, 7.4$; (i) 0.82 or -1.82 ; (ii) $0.82 > x > -1.82$; (iii) $-0.7, x = -0.5$; (iv) 1.56 or -2.56 ; (v) $4, 0.8, 1.44$ or -2.44 ; (vi) 0.62 or -1.62 .
11. $-9.6, -3.6, -1.2, -1.2, -2.4, -3.6, -3.6, -1.2, 4.8$; (i) 4.27 ; (ii) $x > 4.27$; (iii) $-3.83, x = 2.53$; (iv) $-0.974, x = -0.528$; (v) 4.52 ; (vi) (a) $3.78, 0.71, -1.49$; (b) -2.31 ; (c) 4.62 ; (vii) $3, 2, -2$.
12. $1, 1.62, -0.62; 2.20; 0.66$. 13. $2.11, -0.25, -1.86$.
15. $3, 5, -3, 5$.
16. $a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$.

II. d. (p. 38.)

1. $1, 1 \pm \sqrt{\frac{3}{2}}, 1 \pm \sqrt{-\frac{3}{2}}, 1 \pm \sqrt{\frac{3}{2}}, h \pm \sqrt{-\frac{k}{a}}, \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
2. $(1, 0); [1 + \sqrt{\frac{3}{2}}, 0], [1 - \sqrt{\frac{3}{2}}, 0];$ none; $[1 + \sqrt{\frac{3}{2}}, 0], [1 - \sqrt{\frac{3}{2}}, 0]; [b + \sqrt{-\frac{3}{a}}, 0], [b - \sqrt{-\frac{3}{a}}, 0];$
 $[\frac{-b + \sqrt{b^2 - 4ac}}{2a}, 0], [\frac{-b - \sqrt{b^2 - 4ac}}{2a}, 0].$
3. $1 \pm \sqrt{2}; 2, 0; 1; 1 \pm \sqrt{2}; 2, 0; 1; 1 \pm \sqrt{3}, 1 \pm \sqrt{3}; -3 \pm \sqrt{\frac{3}{2}}; \pm \sqrt{\frac{3}{2}}; 3 \pm \sqrt{\frac{3}{2}}.$
4. All; $1 - \sqrt{\frac{3}{2}} < x < 1 + \sqrt{\frac{3}{2}}; b - \sqrt{-\frac{3}{a}} < b + \sqrt{-\frac{3}{a}};$
 all except $1; 1 - \sqrt{\frac{3}{2}} < x < 1 + \sqrt{\frac{3}{2}}.$
6. $2(x - 1)(x - 1); 2[x - (1 + \sqrt{\frac{3}{2}})][x - (1 - \sqrt{\frac{3}{2}})];$

$$\begin{aligned}
& 2[x - (1 + \sqrt{-\frac{3}{2}})][x - (1 - \sqrt{-\frac{3}{2}})] ; \\
& - 2[x - (1 + \sqrt{\frac{3}{2}})][x - (1 - \sqrt{\frac{3}{2}})] ; \\
& a\left[x - \left(h + \sqrt{-\frac{k}{a}}\right)\right]\left[x - \left(h - \sqrt{-\frac{k}{a}}\right)\right] ; \\
& a\left[x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right]\left[x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right].
\end{aligned}$$

7. No real root.
8. Real and distinct, real and distinct, imaginary, real and equal, real and distinct, imaginary.
9. $x^2 - x - 2 = 0$, $4x^2 - 3x = 0$, $x^2 - 2ax + a^2 - b^2 = 0$,
 $x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc = 0$,
 $x^2 - 2 = 0$, $x^2 - 2\sqrt{2} \cdot x + 1 = 0$, $(bx - a)(dx + c) = 0$.
10. $\frac{-b \pm \sqrt{b^2 - ac}}{a}$. 11. $6\frac{1}{4}$. 12. $\frac{25}{16}$, $-\frac{25}{16}$.
13. 3, -0.05 . 15. $-\frac{3}{4} < x < \frac{1}{2}$; $-1 < x < 1.5$.
16. $-\infty < x < -4$ and $+1 < x < +\infty$.
17. $0 < a < 4c$.
18. The function cannot lie between -2 and $+2$.
19. $\frac{1 \pm \sqrt{1 - 4k^2}}{2k}$, $-\frac{1}{2} \leq k \leq \frac{1}{2}$, $-\frac{1}{2}$.
20. -7 , -37 , 84 .

II. e. (p. 42.)

1. 3, 2, 3, $4a^2 + 2$, $b^6 + 2$. 2. 10, 100, 1, 0.1 , 10^{2x} .
3. 3, 0.301 , 1.301 , 0 , $3 \log x$.
4. 3, $(x + h)^2 - 3(x + h) + 5$, $\frac{1}{x^2} - \frac{3}{x} + 5$.
5. $2x + h + 3$, $2x + 3$.
6. 1, $-\frac{1}{1+h}$, -1 . 7. $9x^2 + 5$, 2 . 8. x^6 , $2h^2$.
11. 1, 2 . 12. $\frac{3}{4}$. 13. $\frac{x-1}{x}$.
14. $(x-3)(x-4)$. 15. $8(x-3)$. 16. $\frac{3(x-3)}{x-1}$.
17. $3x^2$ or $x^2 + 2$. 18. $\log x$. 19. 10^x .
20. $x^3 - x$. 21. $(-1)^x$.

III. a. (p. 48.)

1. Yes, *e.g.*, 3000 ; no ; 2.
2. 11, no, 2.
3. Approx. 0.09, 0.01, 0.001, 0.0001 ; 0.
4. - 1, 7. 5. 1.5, 1.67, 1.91, 1.99 ; 2 ; no ; $0.9991 < x < 1.001$.
6. $\frac{1}{2}(h^2 + 3h + 3)$, 1.5.
7. $16(h + 1)$, 16.
8. 5.25, $5 - 2a$, 2.5.
9. $-\frac{1}{4}$, $-\frac{1}{a^2}$, ± 2 .

III. b. (p. 52.)

1. 70 ; 250 ; 1890 to 1900, 390.
2. $\frac{x}{a}$, $\frac{y}{b}$, $\frac{y-x}{b-a}$, $y - x$ miles per min.
3. 8.4, 24.9, 27.3, 29, 37.2 m.p.h.
4. 1.5, 2.6, 2.25, 3.2, 1.3, 2.4 ft. per sec.
5. For $t = 0, 5, 10, 15, 20, 25$, rate is 1.76, 0.88, 0.43, 0.21, 0.11, 0.05 cu. ft. per sec.

III. c. (p. 56.)

1. 0.47, - 2.9, 0.
2. 0.021, - 0.020, 0.021, 0.009, - 0.0077.
4. 0.93, 0.38 ins.
5. 2, 2, 2.
6. - 5.
7. 0.6.
8. 3 ft., 3 ft. per sec. ; 12 ft., 6 ft. per sec., 9 ft. per sec. ; 13.23 ft., 12.3 ft. per sec. ; $3(2 + h)^2$ ft., $12 + 3h$ ft. per sec. ; 12.3, 12.03, 12.000003, 12 ft. per sec.
9. 144 ft., 48 ft. per sec. ; 64 ft., 80 ft. per sec. ; 134.56 ft., 94.4 ft. per sec. ; $16(3 - h)^2$ ft., $96 - 16h$ ft. per sec. ; 94.4, 95.84, 95.999984, 96 ft. per sec.
10. 32 ft. per sec.
11. $32a$ ft. per sec.
12. 3, 2, 0, - 2.
13. 5.1, 5, - 1.5.
14. $-\frac{1}{4}$, $-\frac{1}{a^2}$.
15. $6c^2$.
16. $\frac{3}{4}$, $\frac{1}{2}$.
17. ∞ .
18. m .
19. $4a + b$.
20. $2ax_1 + b$, $-\frac{b}{2a}$.
22. $2x$.
23. 0.

IV. a. (p. 64.)

1. $5 \cdot \delta x$, 5.
2. $2x \cdot \delta x + (\delta x)^2$, $2x$.
3. $10x \cdot \delta x - 3\delta x + 5(\delta x)^2$, $10x - 3$.
4. $(1 + x + \delta x)^2$, $2x + 2$.

5. $1 + 2x$. 6. $2x, 6x, 14x, 200x$.
 7. $3x^2, 12x^2, 15x^2, 87x^2$. 8. $12x^3, 40x^3$.
 9. $35x^4, 5x^4$. 10. $4\frac{1}{2}x^5, -6x^5, 1 - 12x^5$.
 11. $\lim_{\delta x \rightarrow 0} \frac{(x + \delta x)^n - x^n}{\delta x} = nx^{n-1}$. 12. $3x^2 + 4x, 6x + 3x^2, 2x^2 + x$.
 13. $3u + 5v$. 14. 3, 5, 30x, 8, no, yes.
 15. Yes, yes, no, no. 16. $\frac{d}{dx}x^n = nx^{n-1}$.
 17. $\frac{d}{dx}(cx^n) = c\frac{d}{dx}(x^n)$. 18. $\frac{d}{dx}(Ax^m + Bx^n) = \frac{d}{dx}(Ax^m) + \frac{d}{dx}(Bx^n)$.
 20. The differentials of $x^8, x^{50}, x^{-1}, x^{-3}, x, x^0, x^{\frac{3}{2}}, x^{\frac{1}{2}}, x^{-\frac{1}{2}}, x^{-m}$ are
 $8x^7, 50x^{49}, -x^{-2}, -3x^{-4}, 1, 0, \frac{3}{2}x^{\frac{1}{2}}, \frac{1}{2}x^{-\frac{1}{2}}, -\frac{1}{2}x^{-\frac{3}{2}}, -mx^{-m-1}$.
 21. 0.41. 22. $-\frac{1}{30}$. 23. 0.331, 0.030301. 24. 0.008. 25. -20.

IV. b. (p. 68.)

1. $7x^6$. 2. $30x^2$. 3. $1 - \frac{1}{x^2}$. 4. $-\frac{5}{x^6}$.
 5. $-\frac{3}{x^2}$. 6. $-\frac{4}{x^3}$. 7. $6x - 2$. 8. x^3 .
 9. $5x^4$. 10. $375x^2$. 11. $2x - 2$. 12. $-\frac{n}{x^{n+1}}$.
 13. a . 14. ax^{a-1} . 15. $-\frac{3c}{x^4}$. 16. $\frac{b}{2\sqrt{x}}$.
 17. $\frac{3}{2}\sqrt{x}$. 18. $-\frac{1}{2x^{\frac{3}{2}}}$. 19. 0. 20. $\frac{1}{3}x^{-\frac{2}{3}}$.
 21. $2x^2 - \frac{1}{5}$. 22. $\frac{1}{10} - \frac{10}{x^2}$. 23. $21x^2 - 2$. 24. $4\pi x^2$.
 25. $12x^3 - \frac{1}{2} + \frac{6}{x^2}$. 26. $20x^3 + 6x - \frac{1}{x^2} + \frac{2}{x^3}$.
 27. $3x^2 + 4x + 1$. 28. $2x - \frac{2}{x^3}$. 29. $2nx^{2n-1}$. 30. $-kx^{-k-1}$.
 31. 6. 32. 6. 33. 1, 2. 34. $42x$.
 35. $\mp \frac{1}{4}x^{-\frac{3}{2}}$. 39. 68, 100; -32, -32. 40. $392x^7, 392x^7$.
 41. $12 + 20x, 5, 2(3 + 5x)^2, 60 + 100x$.
 42. $4 + 6x^2 + 3x^4, 2x, 2 + 4x^2 + 3x^4 + x^6, 8x + 12x^3 + 6x^5$.

IV. c. (p. 71.)

1. Deer., incr., -, +, yes.
2. Incr., incr., +, +, no.
3. Incr., incr., incr., decr., decr., +, +, +, -, -, yes.
4. Max. C ; min. A, E .
5. Max.; decr., decr.; -, -.
6. Min.; incr., incr.; +, +.
7. No; decr., incr.; -, +.
8. No; incr., decr.; +, -.
9. +, -; -, +; -, -; +, +.
10. Yes.
11. Yes; no.

IV. d. (p. 76.)

1. $x = 1$ min.; $x = -2$ min.; $x = 0.4$ max.; $x = 1$ min., $x = -1$ max.; none; $x = 1$ min.; $x = \frac{3}{4}$ min., not $x = 0$; $x = 1$ min., $x = -\frac{1}{3}$ max.; $x = 3$ min., $x = -1$ max.; none.
2. 40,000 sq. ft.
3. 2 cu. ft.
4. 4 ft.
5. 8 ins.
6. 80,000 sq. ft.
7. $13\frac{1}{2}$ sq. ft.
8. $10\frac{1}{8}$ sq. ft.
9. 18 cu. ins.
10. 0.927.
11. 8 ft.
12. 2.55 cu. ft.
14. 12 cu. ins.

IV. e. (p. 80.)

1. $5(\delta x)^2, 3x(\delta x)^2 + (\delta x)^3.$
2. $x \cdot \delta x + \frac{1}{2}(\delta x)^2, A = \frac{1}{2}x^2, \frac{1}{2}(\delta x)^2.$
3. $\delta t(10 - 0.6t^2), 0.76$ ft.
4. $-\frac{c}{p^2} \cdot \delta p.$
5. $\delta y = \frac{8x}{3l} \cdot \delta x$; length of wire increases 0.05 in.
6. 540 ft.
7. $\delta A = x^2 \cdot \delta x.$
8. $\delta V = \frac{1}{4}\pi x^2 \cdot \delta x.$
9. $\Delta z \simeq -2x\Delta x, \Delta y \simeq \frac{-x}{\sqrt{1-x^2}}\Delta x, \frac{-x}{\sqrt{1-x^2}}.$
10. $\Delta z \simeq -2x\Delta x, \Delta y \simeq \pm \frac{x}{2\sqrt{4-x^2}}\Delta x, \frac{\pm x}{2\sqrt{4-x^2}}.$
11. $\frac{\pm x}{3\sqrt{x^2-4}}.$
12. $3x + 4y = 5.$
13. $3x + 8y = 10.$
14. $\Delta u \simeq \frac{du}{dy}\Delta y, \frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx}.$

V. a (p. 86.)

1. $y = x^4 + c.$
2. $y = \frac{1}{5}x^5 + \frac{1}{2}x^2 + c.$
3. $\frac{dy}{dx} = 2x$, parallel curves.
4. $y = \frac{1}{12}x^4 + ax + b.$

5. $y = \frac{1}{2}x^2 + c$. 6. $y = x^3 - x^2 + c$.
 7. $y = 4x + c$. 8. $y = ax + b$.
 9. $5x^4, y = \frac{1}{5}x^5 + c, y = \frac{3}{5}x^5 + c$.
 10. $8x^7, y = \frac{1}{8}x^8 + c, y = \frac{5}{8}x^8 - 2x + c$.
 11. $-\frac{1}{x^2}, y = -\frac{1}{x} + c, y = -\frac{4}{x} + c$.
 12. $-\frac{4}{x^5}, y = -\frac{1}{4x^4} + c, y = x + \frac{3}{4x^4} + c$.
 13. $\frac{1}{2\sqrt{x}}, y = 2\sqrt{x} + c, y = \frac{1}{2}x^2 + 6\sqrt{x} + c$.
 14. $y = \frac{5}{3}x^3 - \frac{3}{2}x^2 + 9x + c$. 15. $y = \frac{1}{6}x^6 - \frac{1}{3}x^3 + 3x + c$.
 16. $y = \frac{1}{4}x^4 - \frac{1}{2x^2} + c$. 17. $y = \frac{1}{3}x^3 + c$.
 18. $c = -1$. 19. $s = 12 + 50t - 16t^2$.
 20. $s = \frac{2}{3}t^3 + 5t + 8$. 21. 165 ft.
 22. $x = 12t, y = 16t^2; y = \frac{1}{9}x^2; 2\frac{1}{2} \text{ sec.}; 30 \text{ ft.}; \frac{2}{3}0; x = 10,$
 $y = \frac{100}{9}$.
 23. $\frac{dT}{dx} = a(K - T)$. 24. $\frac{dp}{dx} = -ap$.

V b. (p. 92.)

1. $\frac{1}{3}x^3 - 2\frac{1}{3}, \int_1^2 x^2 dx$. 2. $\frac{2}{3}x^{\frac{3}{2}}$. 3. 4.
 4. 1, 4; $4\frac{1}{2}$. 5. $\frac{1}{2}, 0.83$. 7. $7\frac{1}{3}, \frac{3}{32}, \left[-\frac{2}{3}, \frac{19}{24}\right]$.
 8. $\pi y^2, \pi(y + \delta y)^2, \frac{1}{6}\pi x^5, 0.148$. 9. $\frac{1}{27}\pi x^3$.
 10. $\sqrt{(100 - x^2)}, \pi(100 - x^2), \pi(100 - x^2), \delta x, \pi(100x - \frac{1}{3}x^3),$
 $\frac{\pi}{3}(2000 - 300x + x^3), 2090 \text{ cu. in.}$ 11. A sphere; $\frac{4}{3}\pi r^3$.
 12. $2\frac{2}{3}$. 13. $128\pi, 192\pi$. 14. 3050 cu. in.

VI. a. (p. 97.)

1. 12, $2n$; $36, n^2; \frac{1}{13}, \frac{1}{2n+1}; 6 \times 10^6, n \times 10^n; 96, 3 \times 2^{n-1};$
 $\frac{11}{64}, \frac{2n-1}{2^n}; 40, 7n-2; -5, 13-3n; 6 \times 10^{-6}, n \times 10^{-n};$
 $1, (-1)^n; 343, (n+1)^3; 10^6 + 24, 10^n + 4n$.

2. 1, 3, 5; $\frac{1}{6}$, $\frac{1}{2}$, $\frac{3}{2}$; $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$; 2, 6, 12; 8, 40, 315; 0, 7, 26.
 3. 1, 3, 2, 10, n . 4. 1, 16, n^2 . 5. 29, no, $14 \times 29 = 406$.
 6. 29, 435; 35, 385; -15 , -165 ; $9\frac{1}{4}$, $101\frac{1}{4}$. 7. 27, 22, 21, 19.
 8. 104 , $7n - 1$; 128, 2^{n-1} ; 102, $11n - 8$. 9. 15, $3\frac{1}{4}$, 11, 7.
 10. 2866, 23. 11. $\left[\frac{13 \cdot 12 \cdot 11 \cdot \dots \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot \dots \cdot 9 \cdot 10} b^{10} \right] \times$ *this is the 11th term*

VI. b. (p. 100.)

1. $19\frac{1}{2}$. 2. $7\frac{2}{3}$. 3. 87.
 4. $56\frac{1}{2}$. 5. $19\frac{1}{4}$, $26\frac{1}{2}$, $33\frac{3}{4}$. 6. 5029.
 7. -130 . 8. $702 \cdot 5$. 9. $-79\frac{1}{2}$.
 10. $2n^2 + n$. 11. $12a + 66d$. 12. $\frac{1}{2}n[2a + d(n - 1)]$.
 13. $258\frac{1}{2}$. 14. 540. 15. $\frac{1}{2}n(a + l)$.
 16. 29; $\frac{1}{2}(n^2 - n + 2)$. 17. $[4215] 4315$.
 18. $2x - a$; $x(n - 1) + a(2 - n)$. 19. 91 , $n^2 - n + 1$.
 20. 5 , $\frac{n}{2}(5n - 17)$. 21. Yes, no. 22. 999000.
 23. $l - (n - 1)d$. 24. $\left[\frac{1}{2}(b + a)(b - a + 1) \right] - 1$. *Last term.*

VI. c. (p. 103.)

1. $7 \times 5^{n-1}$, $(-2)^{n-1}$, 2^{4-n} , $-10(-0.6)^{n-1}$, $\frac{7^{n-1}}{6^{n-2}}$, $\frac{3^{2n-3}}{7^{n-2}}$,
 $(-1)^n x^{2n-1}$, $\frac{b^{n-1}}{a^{n-2}}$, $\frac{1}{a^{n-2}}$.
 2. $-\frac{1}{18}$, 2. 3. $(\frac{3}{4})^{25}$. 4. 1533.
 5. $80\frac{2}{3}4\frac{2}{3}$. 6. $ar^7 - a$; $\frac{a(r^7 - 1)}{r - 1}$.
 7. $2(3^{10} - 1)$, $\frac{2 \cdot 5}{3}[1 - (0.4)^{12}]$, $\frac{9}{4} - (\frac{2}{3})^6$, $\frac{7}{3}(2^{11} + 1)$, $\frac{2 \cdot 4 \cdot 3}{13}[1 - (\frac{4}{9})^{20}]$.
 8. $1 + r + r^2 + r^3 + r^4 + r^5$; $1 + r + r^2 + \dots + r^9$; $\frac{1 - r^{21}}{1 - r}$, $\frac{a(1 - r^{21})}{1 - r}$.
 9. ar^{n-1} , ar^{n-2} , $ar + ar^2 + ar^3 + \dots + ar^{n-2} + ar^{n-1} + ar^n$, $ar^n - a$,
 $\frac{a(r^n - 1)}{r - 1}$.
 10. 1.111; 0.000001 per cent., $\frac{1}{9}(1 - 10^{-50})$, 10^{-48} per cent.;
 $\frac{1}{9}[1 - 10^{-100}]$, $\frac{1}{9}[1 - 10^{-1000}]$, $\frac{1}{9}[1 - 10^{-1000000}]$;
 $\lim_{n \rightarrow \infty} \frac{1}{10^n} = 0$, $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{10^n}\right) = 1$; $\frac{1}{9}$.

11. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ ins. ; $\frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, 1 - 2^{-10}, 1 - 2^{-100}$ ins. : 1,
 $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$; $2^{-6} - 2^{-46}$; 16.

12. (ii), (iv).

14. $\frac{43}{99}$.

15. 2500, $8\frac{1}{3}$, $\frac{a}{1-r}$.

16. 630, 23600.

VI. d. (p. 106.)

1. 17.6 m.p.h. $2\frac{1}{8}$ m.

2. 0.0966.

3. 36 ft.

4. 65.

5. 200° .

6. n^2 .

7. $\frac{1 - (n+1)x^n + nx^{n+1}}{1-x}$; $\frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2}$.

8. Less than 2^{200} .

10. $\frac{5}{11}$.

11. 1, 7, 19, 37.

VI. e (p. 108.)

1. 1, 1.18, 1.38, 1.63, 1.91 ; 1.18 : 1 in each case. 2. 31.8 millions

3. 46.9° C. 4. £101,000. 5. 40, 1.19. 6. 4.1 per cent.

VI. f. (p. 116.)

1. \$3680.

2. \$976.45.

3. \$12014.15.

4. \$11,111.11.

5. \$100.

6. \$100.

7. 92.64.

8. $\$11,772.64$

9. $\$[(13.0079)x]$.

10. \$10,000.

11. \$250,000.

12. \$53,279.

13. \$6641.11.

14. \$542.85.

15. \$3576.33.

VII. a. (p. 123.)

1. 20, 210, 5040, 30240, 720, 5040.

2. 120.

3. 40320.

4. 151,200, 3628800.

5. 12, 132, 1000, 999000, n , $n^2 - n$, $n^3 - 3n^2 + 2n$,
 $(n+1)(n+2)(n+3) \dots (2n-1)2n$.

6. 3024, 6561.

7. 7.

8. 9000, $\boxed{6561}$ no.

9. 40320.

10. 362880, 40320, 40320, 5040.

11. 576, 14400.

12. 720, $\boxed{1056}$.

VII. b. (p. 127.)

1. 10. 2. 364. 3. 22100. 4. $\frac{52!}{13! 39!}$
 5. 66 ; 220. 6. 13860. 7. 924, 34650.
 8. 1980, 3576. 9. 161,700. 10. 1.
 11. 1, 1, 1. 12. 60.

VIII. a. (p. 130.)

1. $x^4 + x^3(a + b + c + d) + x^2(ab + ac + ad + bc + bd + cd) + x(abc + bcd + cda + dab) + abcd;$
 $x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4; x^4 + 4x^3 + 6x^2 + 4x + 1;$
 $x^4 - 4x^3 + 6x^2 - 4x + 1.$
 2. 64, 6, 15.
 3. $abc + abd + abe + acd + ace + ade + bcd + bce + bde + cde;$
 $10a^3; -10.$
 4. 4, 35, 35.
 5. $x^3 + 6x^2 + 11x + 6; x^4 + 10x^3 + 35x^2 + 50x + 24.$
 6. $6x^3 + 11x^2 + 6x + 1; 24x^4 + 50x^3 + 35x^2 + 10x + 1.$

VIII. b. (p. 131.)

1. $1 + 3x + 3x^2 + x^3.$ 2. $1 + 4x + 6x^2 + 4x^3 + x^4.$
 3. $1 - 4x + 6x^2 - 4x^3 + x^4.$ 4. $x^3 - 6x^2 + 12x - 8.$
 5. $x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6.$
 6. $x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}.$ 7. $8x^3 - 36x^2y + 54xy^2 - 27y^3.$
 8. $a^8 - 8a^6b + 24a^4b^2 - 32a^2b^3 + 16b^4.$ 9. $35, \frac{4}{3}n(n-1)(n-2).$
 10. $-15360x^3.$ 11. 24. 12. 11.
 13. $-20x^3y^3.$ 14. $120x^7y^3, 120x^3y^7.$
 15. ${}^nC_{r-1} \cdot 2^{r-1}x^{r-1}; (-1)^{r-1} \cdot {}^nC_{r-1} \cdot 2^{r-1}x^{r-1}.$
 16. ${}^nC_{r-1}x^{n-r+1}y^{r-1}; {}^nC_{n-r+1}x^{r-1}y^{n-r+1};$ Yes. 17. $x^3.$ 18. $x^4.$
 19. ${}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n - 1$ $1 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \dots = 0.$
 20. 0. 21. ${}^nC_r - {}^nC_{r-1} = \frac{(n-2r+1)|n}{|r|n-r+1}.$
 22. $1 + nx + \frac{n(n+1)}{2} \cdot x^2 + \frac{n(n-1)(n+4)}{6} x^3.$ 23. 34.
 24. 104060401, 9509900499. 25. 1.1041, 0.83297.

VIII. c. (p. 133.)

2. $\frac{x^5}{1-x}$. 3. $1+x+x^2+x^3+x^4$.
 4. 0.01, 0.001. 5. $\frac{x^2}{1+x}, -\frac{x^3}{1+x}, \frac{x^4}{1+x}$.
 6. $1-x+x^2-x^3+x^4$; 0.0001, 0.000001. 7. $\frac{3x^2-2x^3}{(1-x)^2}, \frac{3x^2+2x^3}{(1+x)^2}$.
 8. $1+2x+3x^2+4x^3, 1-2x+3x^2-4x^3$; 0.0003, 0.0003.
 10. $1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3$, 0.00001. 11. $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3$, 0.001.
 12. $\frac{1}{x}-\frac{h}{x^2}$, 0.98. 13. $\frac{1}{x}-\frac{h}{2x^2}$, 0.499, 0.332.

VIII. d. (p. 137.)

1. ab , 0.0002, 1.08. 2. 1.06, $1+3h$. 3. 1.000033; 1, 1.000099; 1.
 4. 2 per cent. 5. 6 per cent. 6. 3, 2; 1.5 per cent.
 7. $1+2h$, 1.02. 8. 1.01, 0.94, 1.003, 1.004, 1.005, 0.1999.
 9. $1+2h, 1-5h$, 0.95. 10. 0.028, $0.555\frac{h}{\sqrt{l}}$.
 11. 2 per cent. decrease, $-\frac{vp_1}{p}$.
 12. $3x^2, 10x^3, -\frac{1}{x^2}, \frac{1}{2\sqrt{x}}$. 13. Decreased about 3 per cent.
 14. 5 per cent. decrease. 15. 1 per cent. increase.

IX. a. (p. 144.)

Note.—Some of the formulæ below are given to a greater degree of accuracy than can be attained by drawing alone.

1. $y = 1.62x - 0.54$. 2. $l = 17.95 + 0.41W$.
 3. $l = 0.12W + 2.1$. 4. $y = 9.79 - 0.033x$; 0.005.
 5. $y = 6.9x + 0.1$; when $x=3, y=21$. 6. $P = 0.0049v^2$.
 7. $Q = 31000D^2$. 8. $R = 600 + 0.9V^2$.
 9. $k = \frac{1.20}{d} - 0.14$. 10. $S = 4.09t^2$.
 11. $y = \frac{12x}{x-7}$. 12. $t = 0.0419d^{1.163}$.
 13. $\delta = 10^{-8} \times 1.24T^4 - 90$.

REVISION EXERCISES.

A.

1. 2.
3. 1.414, 1.189, 1.090, 1.022, $2^\circ = 1$.
4. 3.162, 1.778, 1.334, 1.154, $10^\circ = 1$.
5. 1.02, 1.01, 0.99, 0.99, 0.995, 1.01, 0.998, 0.098, 10.01, 0.0991.
6. $3x^2$, $\frac{3}{2}x^{\frac{1}{2}}$, $-2x^{-3}$, $\frac{1}{2}x^{-\frac{1}{2}}$, $-\frac{5}{x^6}$, $-\frac{1}{2\sqrt{x^3}}$, 0, $\frac{5}{3}\sqrt[3]{x^2}$.
7. $\frac{1}{3}x^3$, $\frac{2}{3}x^{\frac{3}{2}}$, $-\frac{1}{x}$, $\frac{3}{4}x^{\frac{4}{3}}$, $-\frac{2}{x^2}$, $\frac{2}{2}x^{\frac{5}{2}}$, $\frac{1}{2}x^2$, x , $\frac{3}{2}x^{\frac{2}{3}}$.
8. 1.
9. 3, no finite root.
10. 6.

B.

2. 1.15, 0.7, 4.6.
3. 2.3.
5. $10^{0.4741}$, $10^{-0.0458}$, $10^{0.4283}$.
7. 0.854, 0.854.
8. 0.51, 0.51.
9. $10^{1.301}$, $10^{2.301}$, $10^{3.301}$, $10^{6.301}$
1.301, 2.301, 3.301, 6.301.
10. 1, 3, 0, 5, 5, 3, 4, 5, 7, 8.

C.

2. 18.7.
3. 1.06.
4. 2.08, 4.48, 9.65, 20.8.
5. 8.009.
6. 8.009.
7. 0.0980.
8. 8.009.

D.

2. 0.103.
3. 0.6585.
4. 0.0994.
5. 11.5.
6. 338.
7. 15.6.
8. \$294.
10. 0, $\log x$.
11. $xy = 5$, $xy^3 = 100$, $3^x = 1000$, $x^3 = 10y^2$, $5^x = 6^y$, $5^x = 100 \times 6^y$.
12. 13, 2, -1.
13. 1.06.

E.

1. No.
2. $w = \frac{8P - W}{n}$.
3. 2, 3, 4.
4. Hurdles 25, 20, 15, 10, 5 ;
area 1000, 1600, 1800, 1600, 1000 ; 30.
5. 2.
6. 27.
7. 34, 30, 16.
8. 0.3, 12 ; — 10 ; effort insufficient to start machine.
9. 4.8 in., 19.6 *rt* lb.
10. $\frac{x}{3}$; $\frac{x}{6}$ hrs. ; $2x$ min., $\frac{x}{2}$ hrs.
11. $l + b = 43$.
12. $-\frac{1}{x(x+h)}$, $-\frac{2x+h}{x^2(x+h)^2}$, -1 , -2 .
13. $-1, 4\frac{1}{2}$.

F.

1. 25.2, 5.69. 2. 1800000, 0.018, 0.00018, 1800, $10^4 \times 7.38$.
3. - 16, 4. 4. 1.5. 5. $\frac{Wb}{W - 2Pe}$.
6. $P = \frac{K^2}{a^2} \cdot \sqrt{H^5}$. 7. 1, 7, 1, 5, -2, -2. 8. 1.40.
9. 2, $\frac{1}{512}$, 4, $\frac{1}{256}$, 1. 10. $2\frac{1}{4}$, 1.31 or 0.19. 11. $\frac{1}{4}n^2(n + 1)^2$.
12. 2 or $-\frac{4}{3}$; 1.90 or - 1.23. 13. 50000.
14. $(10x + 5)^2 = 100x(x + 1) + 25$.
15. 2.12 in., $G = 0.743d^{\frac{4}{3}}H^{\frac{1}{3}}$.
16. $(20x - 2x^2)$ sq. in., $AP = 5$ in.
17. 0.25, $T . 75$, $T . 4$, $T . 6667$. 18. 0.36 per cent.
19. 20, 7.5 mm. 20. 10.6 ft.
21. 2.83, 4.24, 1.41, 3.46. 22. 25° with keel.
23. 4, $\frac{1}{4}$, - 2, $-\frac{1}{4}$. 24. 4×10^{15} , 10^{18} .
25. Logarithms 3, - 2, 1.5, 0.25.
26. $\frac{81}{100}$, 9 (8.7).

G.

1. 2.25. 2. $R = \sqrt{\left(\frac{Wr}{17.6P}\right)}$, 2.77.
3. 6750 tons. 4. 1.8 sec. 5. 2.

6. 254 amp. 7. $6, \frac{4}{35}$. 8. 37.3 c.c.
 9. 4500 days. 10. $x = 0.8, 0.6, 0.12$; $y = 12, 24, 120$.
 11. 5.12 : 1. 12. 1.6, 4.4.

H

1. 5, 6; (a) and (b) large and positive; $5 < x < 6$, ≥ -1 (actually $\geq -\frac{1}{4}$).
 2. 1007, -995; 6.0009, 5.9991.
 3. -2, 3, 6, 7, 6, 3, -2; 7; 2; $-0.65 < x < 4.65$; 3.73 or 0.27; 1 or 3; 4.83 or -0.83.
 4. 2.5 dynes. 5. 0, no, large.
 6. 321 cu. in. 7. 2.36, 0.17, -2.53.
 8. 1, $1 \rightarrow \infty, \rightarrow -\infty, x = 0$.
 9. 35800. 11. 2.154.
 12. 2, -6, $4a^2 + 14a - 6, x^2 + 9x + 2$.
 13. $9x^2 + 9xh + 3h^2, 9.01$.
 14. -9, 99, -99.99; no; 100.01, 1000.0001.
 15. 3, $-3.006 < x < -3, -3 < x < -2.995, 1, 1$.
 16. $\frac{2}{25}, \frac{9}{100}, \frac{3}{25}$. 17. 37.5, 20.25, 45, 30 ft. per sec.
 18. $x = \sqrt[3]{175} \cdot y^{\frac{2}{15}}, 3.23$. 19. 2, 3.9, $4 - h, 4$.
 20. 7, 7; $2x + h$ if $h \neq 0$; $2x$.
 21. -4; -1; $-3 < x < 1$; 1.83 or -3.83; 1.24 or -3.24; 0.73 or -2.73.
 22. -4; -3; $-h - 2$ if $h \neq 0$; -2.
 25. 1.15; 0.308; $0 < x < 2$ and $x < -2$; 2.214, -0.539, -1.675; (a) and (b) 2.115, -0.254, -1.861; (c) 1.861, 0.254, -2.115.
 26. 0.8. 28. 1, $\sqrt{2}, 2^x$.

I.

1. $\frac{17}{500}, \frac{1}{25}$, about 0.03.
 2. $2x + 4, -\frac{1}{(x-1)^2}$, 3. -2, $3a^2 - 5$.
 5. -1, -1, 0.
 6. $4 - 6x, 4 - 6x, 4 - 6x, \frac{3}{2} + \frac{4}{5}x, -\frac{6}{x^2}, \frac{4}{\sqrt{x}}$.

7. 18 ; $(-\frac{5}{6}, 4\frac{1}{12})$. 8. $y = 3x - 8$.
9. $8x - \frac{1}{x^2}$, $48x^2 - 8x + 2$, $\frac{5}{3}x^{\frac{2}{3}} - \frac{1}{3}x^{-\frac{4}{3}}$.
10. $3a^2 - 12a - 15$; $5, -1$. 11. $18, 66$ ft. per sec.
12. $-\frac{2x}{(x^2 - 2)^2}$, $\frac{1}{2}$. 13. $\sqrt{\frac{3}{2x}}, \frac{3}{\sqrt{x}}, -\frac{12}{x^4}, 2x - \frac{2}{x^3}$.
14. $0, \pm 2\sqrt{3}$; ± 2 . 15. $4 - \frac{4}{t^2}, 9t^2 + \frac{4}{t^2} - \frac{15}{t^4}$; $12 \cdot 5t^{\frac{3}{2}}$.
16. $3x^2 - 27$; ± 3 ; $54, -54$.
17. Grows lighter and weight vanishes when $x = 7 \cdot 5$.
18. $2x + 3 + h, 2, 2x + 3, 2$.
19. $x = 1$ (min.), $x = -3$ (max.).
21. $4x - x^2, x = 2$. 22. $-3 - 6x^2$.
23. 2 (min.), -2 (max.). 24. $\frac{5}{9}$.
25. $9x^2 - 7, 18x$; $10x, 10$; $7, 0$; $0, 0$; $-\frac{6}{x^2}, \frac{12}{x^3}$.
26. ± 125 . 27. Not if $x = 1$, no, 2.
28. $3400, 3564, 3417 \cdot 84$; $164, 178 \cdot 4$; $180 - 16h$; 180 ft. per sec.
29. $40, 16875$. 30. $x\sqrt{(25 - x^2)}, 12\frac{1}{2}$ sq. cm.
31. $\frac{1}{100}, \frac{7}{1000}, \frac{2+h}{300}, \frac{1}{150}$.
32. $0 \cdot 87$ cu. ft. 33. 165 .
34. $0 \cdot 00101, 0 \cdot 00107, 0 \cdot 00112, 0 \cdot 00107$.
35. $2x + \frac{1}{x^2}, \frac{2}{3}\sqrt[3]{\frac{2}{x}}$. 36. $102, \rightarrow \infty$.
37. $b = 60 + 0 \cdot 3r, 72$. 38. $6 \cdot 955$ in., $y = 2\sqrt{(\pi x)}$.
39. Positive if $0 < x < 4 \cdot 76$, increasing if $x < 3$; $x = 3$.
40. $0 \cdot 73, 11 \cdot 9, 84 \cdot 0$. 41. Each $\simeq 0 \cdot 43, \frac{d}{dx} \log x \simeq \frac{0 \cdot 43}{x}$.
43. $60, 240, 540$; $30, 120, 270$; $20, 80, 180$.
44. Each $\simeq 2 \cdot 3, \frac{d}{dx}(10^x) \simeq 2 \cdot 3 \times 10^x$.
45. $-\frac{5}{x^2}, -\frac{x^2}{5}$. 46. $2x + h - 3, 2x - 3, 1 \cdot 5$.
47. 75 ; $1 \cdot 2, -1 \cdot 2$; 30 (min.), -30 (max.).

J

1. (ii) -20000 . (iii) No. (iv) $3\cdot000$. (v) $\frac{9}{(x-1)(x-5)}, 3$.
2. $y = \frac{1}{6}(3x^2 - 4x^{\frac{3}{2}} + 7), y = 4x + 1 + \frac{1}{2x}$.
3. $\frac{5}{3}x^3 - \frac{4}{x} + c; \frac{5}{11}x^{2\cdot2} - \frac{5}{x^{0\cdot2}} + c$.
4. 24. 5. $135:64$. 6. $\pm 3, \pm 4; 5, -5; 4\sqrt{2}, -4\sqrt{2}$.
7. $x > 15; 0 < x < 3$. 8. $4, -4; x = 3: 10^{\frac{2}{3}}$.
9. $1, 2; 2, 2; 3, 3$.
10. Indices are $0\cdot0791, 2\cdot3980, \bar{2}\cdot1761, 0\cdot8060$.
11. $\pm \frac{2\sqrt{3}}{9} = \pm 0\cdot385$. 12. 170 ft.-lb.
13. 441 mm. 14. $-\frac{1}{4}, 3$. 15. $z = \frac{60x}{y}$.

K.

1. 1, 46; 6, 132; 0, 99.
2. 11, 20, 26, 29; 44; $3n - 1$. 3. 26, 1326.
4. 15 m.p.h., $10\frac{1}{2}$ sec. 5. 2·50. 6. $2^n + n$.
7. $2 \times 3^{n-1}, 3^n - 1$. 8. 3·59. 10. 2, 8, 14.
11. $3^4 - n, 10^{-8} \times 2\cdot3$. 12. 6, 56. 13. $3 \times 2^n, n + 2 + 2^n$.
14. 1973 ft. per sec. 15. 157.
16. $\frac{4}{5}[1 - (-1)^n], 0\cdot8$. 17. $1, \frac{1}{3}, \frac{4}{165}, 12\frac{115}{333}, 12\frac{57}{165}$.

L.

1. ${}^nP_r = \frac{n!}{(n-r)!}, 380$. 2. 1·435.
3. Readings for $w = 35, 50; l = 21 + \frac{w}{3}$.
4. $2n(n-1), 36$. 5. $R = \frac{Av^2}{800}, v \propto \frac{1}{\sqrt{A}}$.
6. 30. 7. $0, \pm 10; 10\cdot005$. 8. 5 per cent.
10. $3\frac{3}{8}$ pints. 11. $12, \frac{3252h}{l\sqrt{l}}, \text{down}$.

12. $a = 1.115, n = 0.5.$

15. Increased 7 per cent.

16. $\frac{h^2}{2x}.$

17. Increased in ratio 1 : 4.

18. 64, 15.

19. 7019298.

M.

1. $P = 0.183W + 0.32.$

3. $2x - 1, -\frac{5}{x^2}, \frac{x^3}{3} + \frac{1}{x} + c, 2\frac{2}{3}.$

4. 2.80.

5. $10^5 \times 7.03, 10^5 \times 7.09.$

6. 4.52 in.

7. $d = 0.11V^2.$

8. $v = 8\sqrt{h}.$

9. 0; +, +; no; -, +.

10. $\frac{1}{30}.$

11. $4\frac{1}{2}.$

12. $h = \frac{P}{0.0092P + 0.29} = \frac{3.4P}{1 + 0.031P}.$

14. Area of quadrant of circle.

15. $t = 0.28d^{1.16}, 232 \text{ sec.}$

16. $\frac{x}{12}(2x^2 - 9x + 12).$

17. $x^3 - x^2 - 4 = 0, \text{ one.}$

18. $d = 11.4\sqrt{H}.$

$$5 \times 5 \times 4 \times 3 \times 2 =$$

$$5 \times 6 \times 6 \times 6 \times 6 \times 5 =$$

$$\begin{array}{r} 36 \\ 36 \\ \hline 216 \\ 216 \\ \hline 1296 \\ 1296 \\ \hline 32400 \end{array}$$

Greek Alphabet.

α	Alpha	χ	chi.
β	Beta	ψ	psi
ϵ	Epsilon	ω	omega.
ζ	zeta		
η	Eta		
θ	Theta		
ι	Iota		
κ	Kappa.		
λ	Lambda		
μ	mu		
ν	nu		
ξ	xi		
$\omicron$	Omicron		
π	pi		
ρ	rho		
σ	Sigma		
τ	Tau		
υ	Upsilon		
ϕ	phi		

[illegible]

F255

O

70. 1. 4.

$$\frac{20}{(100-y)} = \frac{1}{2}$$

$$y = \frac{1}{2} x^2$$

$$\frac{x}{20} = \frac{y}{(100-y)}$$

$$20y = 100x - xy$$

$$20y + xy = 100x$$

$$20 \cdot \frac{1}{2} x^2 + 100x - x \cdot \frac{1}{2} x^2 = 100x$$

$$10x^2 + 100x - \frac{1}{2} x^3 = 100x$$

$$10x^2 - 200x - \frac{1}{2} x^3 = 0$$

$$x^2 + 200x - 20x^2 = 0$$

$$x^2 - 20x + 200 = 0$$

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